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Sylvie Doutre, Marie-Christine Lagasquie-Schiex, Jean-Guy Mailly, Antonio Yuste-Ginel

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April 2025

Incomplete Higher-Order Abstract Argumentation Frameworks

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Abstract

We propose the combination of two known extensions of Dung's abstract Argumentation Frameworks (AFs): Incomplete Argumentation Frameworks (IAFs), where the existence of arguments and attacks may be uncertain, and Higher-Order Argumentation Frameworks (HOAFs), where attacks can be directed not only towards arguments but also towards attacks. As a result, we obtain Incomplete Higher-Order Argumentation Frameworks (IHOAFs), where attacks on attacks can also be uncertain. We take some important initial steps in the formal understanding of this novel framework by providing: (i) the definition of a conservative notion of completion, the hypothetical removal of uncertainty used to reason about argument acceptability in other formalisms for incomplete argumentation; (ii) complexity results for argument acceptability problems; (iii) an analysis of the previous two points in a well-behaved variant of dynamic logic.

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1 Introduction

Formal argumentation is a well-established, interdisciplinary field of research at the intersection of computer sciences, mathematics, linguistics and philosophy. The mathematical modelling of argumentative phenomena developed in this field is deeply influenced by Dung’s Abstract Frameworks (AFs) [18]: these are directed graphs where nodes stand for arguments and edges stand for an attack relation among them. This way of approaching argumentation is *abstract*, as one ignores the nature, internal structure and origin of arguments and their interactions in order to focus on more general, dialectical aspects. In this vein, different semantics are used for selecting *extensions* from a given AF, i.e. sets of arguments considered jointly acceptable because they satisfy some intuitive requirements.

One of the reasons for the extreme popularity of Dung’s framework is probably its simplicity. However, this feature also renders the approach unsuitable for dealing with more refined argumentative notions that, beyond attacks and together with them, have a crucial impact on arguments’ acceptability. This is why, after Dung’s work, the research community developed a well-nourished bunch of extensions of his model. Such extensions can be organised into, at least, two big categories. The first one concerns *the addition of new kinds of interactions between arguments*: for instance, *support relations* (representing a “positive” interaction between arguments; [31, 10, 33]), *higher-order frameworks* (the targets of the interactions can be other interactions, and not only arguments; [7, 5, 11]), or *collective interactions* (the source of the interactions can be a set of arguments and not only a single one; [29]). The second category consists in *the addition of uncertainty to the model*, either by the use of *weights and preferences over arguments and interactions* [2, 6, 35], or by considering the uncertainty about the *presence of the different elements* (both in a qualitative [27, 9] and in a probabilistic fashion [22]). Argumentation frameworks with qualitative uncertainty are particularly useful in applications where they “may arise as intermediate states in an elicitation process, or when merging different beliefs about an argumentation framework’s state, or in cases where complete information cannot be obtained” ([9]).

The intense interest that these extensions of AFs have awakened among formal argumentation practitioners is witnessed by the independent and synchronous study of the combination of two of these models, taking into account both support relations and qualitative uncertainty, leading to the so-called *Incomplete Bipolar Argumentation Frameworks* (IBAFs) [20, 24]. In this paper, we pursue a different extension of AFs, suitable for the simultaneous modelling of higher-order attacks (attacks on attacks) and qualitative uncertainty, resulting in *Incomplete Higher-Order Argumentation Frameworks* (IHOAFs). Our medium-term objective is to extend IHOAFs by incorporating bipolarity, possibly followed by collective interactions, to develop a general argumentation framework that addresses qualitative uncertainty. This contribution represents a first step toward that goal.

Let us start with illustrating the kind of situation that motivates the devel-

opment and study of IHOAFs through the following simple example:

Example 1 *A discussion takes place between friends to decide what to do during the day:*

a : “We should go to the beach this afternoon because it is sunny.”

b : “Sam told me that it will rain later.” (b attacks a)

c : “Yes, but it will not last according to yesterday’s weather forecast.” (c attacks the attack from b to a)

Argument c attacks neither a nor b, but the fact that b attacks a: this is an attack on an attack. An extension of Dung’s framework with higher-order attacks (HOAF) is required to model such an example. Moreover, given how imprecise the weather forecast can be (even more when it is yesterday’s forecast), argument c may be uncertain and disregarded. In addition, even short, a rain may prevent any outing to the beach, so the attack from c to the attack from b to a may be disregarded. Such a situation regarding argument c and the attack it is at the source of, requires the addition of uncertainty regarding the presence of elements in the model, that is, an IHOAF. \square

Following the presentation of the background (see Section 2), this paper provides the first study of IHOAFs through several contributions: (i) in Section 3, we formally define IHOAFs and introduce a notion of completion (the hypothetical removal of uncertainty in incomplete frameworks to facilitate reasoning about argument or interaction acceptability); (ii) also in Section 3, we analyze the computational complexity of reasoning about arguments and attacks acceptability problems in IHOAFs; (iii) in Section 4, we encode these problems using the Dynamic Logic of Propositional Assignments (DL-PA), a well-behaved variant of propositional dynamic logic that has recently proven effective for reasoning about argumentation. Finally, Section 5 provides a discussion and conclusion, outlining future perspectives. Proofs are included in the Appendix.

2 Background

2.1 Abstract Argumentation Frameworks without Uncertainty

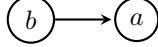
Let \mathbf{A} be an infinite countable set of arguments. The main definition of an *Abstract Argumentation Framework* has been proposed by Dung in [18]:

Definition 1 (Basic Abstract Argumentation Framework - AF) *An Abstract Argumentation Framework (AF) is a pair $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ with $\mathcal{A} \subseteq \mathbf{A}$ the finite set of arguments and $\mathcal{R} \subseteq \mathcal{A} \times \mathcal{A}$ the set of attacks. For $a, b \in \mathcal{A}$, we say that a attacks b if $(a, b) \in \mathcal{R}$ (and we sometimes use the infix notation*

$a\mathcal{R}b$).

If $(a, b) \in \mathcal{R}$ and b attacks some $c \in \mathcal{A}$, then a defends c against b . Similarly, a set $S \subseteq \mathcal{A}$ attacks (resp. defends) an argument b if there is some $a \in S$ that attacks b (resp. if, for any $a\mathcal{R}b$, there is $c \in S$ that defends b against a).

Let us consider Example 1 without taking into account argument c : arguments b and a and their relationship can be represented by the following directed graph (the arguments are given in round nodes and the simple plain arrow represents the attack from b to a):



We classically use the concept of *extensions*, proposed by Dung [18], for evaluating the acceptability of arguments, i.e. sets of collectively acceptable arguments. The usual semantics are based on two main principles: conflict-freeness and admissibility.

Definition 2 (Conflict-Freeness and Admissibility in AFs) Given $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ an AF, the set $E \subseteq \mathcal{A}$ is conflict-free iff $\forall a, b \in E, (a, b) \notin \mathcal{R}$; $a \in \mathcal{A}$ is acceptable w.r.t. E iff E defends a ; E is admissible iff it is conflict-free and $\forall a \in E, E$ defends a .

Four main semantics are proposed by Dung (see more details in [18, 4]):

Definition 3 (Extensions for an AF) Given $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ an AF. Let $E \subseteq \mathcal{A}$ be an admissible set of \mathcal{F} . E is:

- a complete extension for \mathcal{F} iff E contains all the arguments that it defends;
- a preferred extension for \mathcal{F} iff E is a \subseteq -maximal admissible set;
- a grounded extension for \mathcal{F} iff E is a \subseteq -minimal complete extension;
- and a stable extension for \mathcal{F} iff $\forall a \in \mathcal{A} \setminus E, E$ attacks a .

We use $\text{co}(\mathcal{F})$, $\text{pr}(\mathcal{F})$, $\text{gr}(\mathcal{F})$ and $\text{st}(\mathcal{F})$ for the sets of (resp.) complete, preferred, grounded and stable extensions of \mathcal{F} .

Let $x \in \mathcal{A}$ and $\sigma \in \{\text{co}, \text{pr}, \text{gr}, \text{st}\}$ be a semantics. We say that x is credulously (resp. skeptically) accepted by σ if and only if $\exists E \in \sigma(\mathcal{F})$ such that (resp. $\forall E \in \sigma(\mathcal{F})$) $x \in E$.

2.2 Incomplete Argumentation Frameworks

Incomplete AFs are AFs with qualitative uncertainty about the presence of some arguments or attacks [13, 9, 8, 27]. Formally:

Definition 4 (Incomplete Argumentation Framework - IAF) An Incomplete Argumentation Framework (IAF) is a tuple $\mathcal{I} = (\mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^?)$ where:

- $\mathcal{A} \subseteq \mathbf{A}$ is the finite set of certain arguments;

- $\mathcal{A}^? \subseteq \mathbf{A}$ is the finite set of uncertain arguments;
- $\mathcal{R} \subseteq (\mathcal{A} \cup \mathcal{A}^?) \times (\mathcal{A} \cup \mathcal{A}^?)$ is the set of certain attacks;
- and $\mathcal{R}^? \subseteq (\mathcal{A} \cup \mathcal{A}^?) \times (\mathcal{A} \cup \mathcal{A}^?)$ is the set of uncertain attacks.

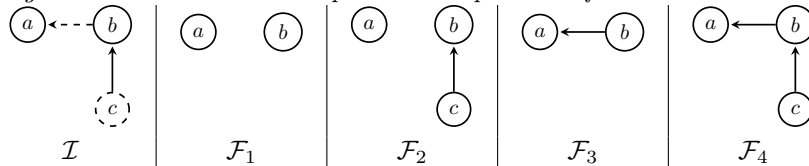
\mathcal{A} and $\mathcal{A}^?$ are disjoint sets of arguments, and \mathcal{R} , $\mathcal{R}^?$ are disjoint sets of attacks.

Intuitively, \mathcal{A} and \mathcal{R} correspond, respectively, to arguments and attacks that certainly exist, while $\mathcal{A}^?$ and $\mathcal{R}^?$ are those that may (or may not) exist. In a multi-agent, adversarial perspective, \mathcal{A} and \mathcal{R} can be understood as the arguments and attacks that an agent knows her opponent is aware of; while $\mathcal{A}^?$ and $\mathcal{R}^?$ are the arguments and attacks such that the agent does not know whether her opponent entertains. This way of modelling uncertainty has a tight connection to some simple epistemic logics of knowing whether (see [21]). Note that a certain attack can exist between two uncertain arguments (these are sometimes called *conditionally certain* attacks); that means that, if the agent is *aware of these arguments*, then the attack is certain. Reasoning about IAFs is usually made through the notion of completion, i.e. a classical AF that represents a “possible world” w.r.t. the uncertain information encoded in the IAF. Formally:

Definition 5 (Completion for an IAF) Given $\mathcal{I} = (\mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^?)$ an IAF, a completion of \mathcal{I} is an AF $(\mathcal{A}_c, \mathcal{R}_c)$ such that $\mathcal{A} \subseteq \mathcal{A}_c \subseteq \mathcal{A} \cup \mathcal{A}^?$ and $\mathcal{R} \cap (\mathcal{A}_c \times \mathcal{A}_c) \subseteq \mathcal{R}_c \subseteq (\mathcal{R} \cup \mathcal{R}^?) \cap (\mathcal{A}_c \times \mathcal{A}_c)$.

Example 2 Consider $\mathcal{I} = (\mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^?)$ depicted in column 1 of the figure below, where $\mathcal{A} = \{a, b\}$ (plain nodes), $\mathcal{A}^? = \{c\}$ (dashed node), $\mathcal{R} = \{(c, b)\}$ (plain edge) and $\mathcal{R}^? = \{(b, a)\}$ (dashed edge). It means that the arguments a and b certainly exist, and there is uncertainty regarding the existence of the attack (b, a) . Moreover, the argument c is uncertain, but if it exists then the attack (c, b) also exists.

Figures in columns 2 to 5 depict the completions of \mathcal{I} .



\mathcal{F}_1 shows the situation where no uncertain elements actually exist, while \mathcal{F}_4 shows the opposite situation (all the uncertain elements appear). \mathcal{F}_2 and \mathcal{F}_3 show the intermediate situations, where only one uncertain element (either the argument c , or the attack (b, a)) exists. \square

Reasoning tasks like credulous (or skeptical) acceptance or verification are thus redefined over completions [9, 8, 27]. Hence, each classical task has two variants: the possible view (the property holds in some completion) and the necessary view (the property holds in each completion). These reasoning tasks are, in most cases, computationally harder than their counterpart for standard

AFs (under the usual assumption that the polynomial hierarchy does not collapse) [9, 8]. This can be explained by the exponential number of completions. For instance, the acceptance problems for the grounded semantics are NP or coNP-complete in the case of IAFs whereas it is P-complete for AFs (see [8, 27] for more details).

2.3 Higher-Order Argumentation Frameworks

Higher-Order Argumentation Frameworks introduced by Barringer et al [7] are argumentation frameworks in which attacks can target other attacks. To define such a framework, we need to consider that \mathbf{A} corresponds to an infinite countable set of *argument names*, whose elements are denoted using small-case Latin letters; and to suppose the existence of an infinite countable set of *attack names* \mathbf{R} , disjoint with \mathbf{A} , whose elements are denoted using small-case Greek letters.

Note that, since attacks are now identified from names, we must determine their source and their target and we also must establish the relation between an attack and the couple of its source and target using the following notation:

Notation 1 Let $\mathcal{R} \subseteq \mathbf{R}$. Let $\alpha \in \mathcal{R}$. $\mathbf{v}(\alpha)$ denotes the pair giving the source and the target of α . For $X \subseteq \mathcal{R}$, $\mathbf{v}(X) = \{\mathbf{v}(\alpha) \mid \alpha \in X\}$.

Then the basic definition for Higher-Order Argumentation Frameworks is the following:

Definition 6 (Higher-Order Argumentation Framework - HOAF)

A Higher-Order Abstract Argumentation Framework (HOAF) is a tuple $\mathcal{H} = (\mathcal{A}, \mathcal{R}, \mathbf{src}, \mathbf{trg})$ such that:

- \mathcal{A} is a finite set of argument names ($\mathcal{A} \subseteq \mathbf{A}$).
- \mathcal{R} is a finite set of attack names ($\mathcal{R} \subseteq \mathbf{R}$).³
- $\mathbf{src} : \mathcal{R} \rightarrow \mathcal{A}$ is the function that assigns a source to each attack name.
- $\mathbf{trg} : \mathcal{R} \rightarrow \mathcal{A} \cup \mathcal{R}$ is the function that assigns a target to each attack name.⁴

Moreover, $\forall \alpha, \beta \in \mathcal{R}, \alpha \neq \beta$ implies $\mathbf{v}(\alpha) \neq \mathbf{v}(\beta)$.^{5,6}

Example 3 To illustrate the previous definition, let us consider the following HOAF (argument names are given in round nodes whereas attack names are

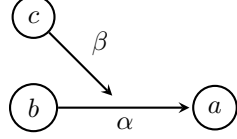
³Since \mathbf{A} and \mathbf{R} are disjoint, we also have $\mathcal{A} \cap \mathcal{R} = \emptyset$.

⁴Note that this definition allows self-attacking attacks (*i.e.* attacks α such that $\mathbf{trg}(\alpha) = \alpha$) even if the intuition behind this case is not clear.

⁵Note that, following the previous definition and the notation of \mathbf{v} , we have: $\forall \alpha \in \mathcal{R}, \mathbf{v}(\alpha) = (\mathbf{src}(\alpha), \mathbf{trg}(\alpha))$.

⁶This constraint avoids the fact that two attacks with different names could have the same source and the same target. This has been evoked in [12] as the possibility of having similar attacks in different contexts. Here we do not consider this possibility.

the labels on the corresponding arrows) which corresponds to the framework of Example 1 given in the introduction (the discussion about what to do today), without the uncertainty about the presence of elements:



Here we have three arguments (a , b and c) and two attacks (α and β). α is a first-order attack (it targets the argument a) whereas β is a second-order attack (it targets α).

□

This kind of framework has been studied from different angles: Extended Argumentation Frameworks (EAF) by Modgil [28]; Argumentation Frameworks with Recursive Attacks (AFRA) by Baroni et al. [5]; and Recursive Argumentation Frameworks (RAF) by Cayrol et al. [12]. Each approach has produced appropriate but different semantics. In this paper, we only consider the RAF case, leaving the other cases for future work.

Within the RAF approach, semantics are based on the notion of *structure*, which replaces the extensions of [18], and extended versions of acceptability and conflict-freeness (see [12]).

Definition 7 (Acceptability and Conflict-Freeness in RAF) Let $\mathcal{H} = (\mathcal{A}, \mathcal{R}, \mathbf{src}, \mathbf{trg})$ be an HOAF. Let $x \in \mathcal{A} \cup \mathcal{R}$, $S \subseteq \mathcal{A}$ and $\Gamma \subseteq \mathcal{R}$. Let $U = (S, \Gamma)$. U is called a structure of \mathcal{H} and:

- U defeats x iff $\exists \alpha \in \Gamma$ such that $\mathbf{src}(\alpha) \in S$ and $\mathbf{trg}(\alpha) = x$.
- x is acceptable w.r.t. U iff $\forall \alpha \in \mathcal{R}$, if $\mathbf{trg}(\alpha) = x$, then $\exists \beta \in \Gamma$ such that $\mathbf{src}(\beta) \in S$ and either $\mathbf{trg}(\beta) = \alpha$ or $\mathbf{trg}(\beta) = \mathbf{src}(\alpha)$. $\mathbf{Acc}(U)$ is the set of acceptable elements w.r.t. U .
- U is conflict-free iff $\nexists \alpha \in \Gamma$ such that $\mathbf{src}(\alpha) \in S$ and $\mathbf{trg}(\alpha) \in S \cup \Gamma$.

Structure-based semantics are defined as shown below (see [12]).

Definition 8 (Semantics for RAF) Let $\mathcal{H} = (\mathcal{A}, \mathcal{R}, \mathbf{src}, \mathbf{trg})$, $U = (S, \Gamma)$ be a structure of \mathcal{H} .

- U is complete iff U is conflict-free and $\mathbf{Acc}(U) = S \cup \Gamma$.
- U is preferred iff U is a \subseteq -maximal conflict-free structure such that $S \cup \Gamma \subseteq \mathbf{Acc}(U)$.
- U is grounded iff U is the \subseteq -minimal complete structure.
- U is stable iff U is conflict-free and defeats any element $x \in (\mathcal{A} \cup \mathcal{R}) \setminus (S \cup \Gamma)$.

Let $\sigma \in \{\text{co}, \text{pr}, \text{gr}, \text{st}\}$ be a semantics, $\sigma(\mathcal{H})$ denotes the set of structures of \mathcal{H} .

Note that the following proposition holds:

Proposition 1 *Let \mathcal{H} be an HOAF. The preferred structures of \mathcal{H} are its \subseteq -maximal complete structures and the grounded structure is unique.*

Example 4 *In the framework of Example 3, $(\{a, b, c\}, \{\beta\})$ is the unique complete, preferred, grounded and stable structure. Note that, even if a is attacked by b , a and b can be accepted together since this attack is invalidated by β . \square*

3 Incomplete HOAFs: Definitions and Complexity

A first step for defining incompleteness for enriched AFs has been done by [20, 24] considering only the addition of a support relation whose elements might be uncertain. Here, we want to take into account qualitative uncertainty together with another kind of enrichment: higher-order attacks.

Definition 9 (Incomplete HOAF - IHOAF) *An Incomplete Higher-Order Argumentation Framework (IHOAF) is a tuple $\mathcal{IH} = (\mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^?, \mathbf{src}, \mathbf{trg})$ where:*

- \mathcal{A} is the finite set of certain argument names ($\mathcal{A} \subseteq \mathbf{A}$).
- $\mathcal{A}^?$ is the finite set of uncertain argument names ($\mathcal{A}^? \subseteq \mathbf{A}$).
- \mathcal{R} is the finite set of certain attack names ($\mathcal{R} \subseteq \mathbf{R}$).
- $\mathcal{R}^?$ is the finite set of uncertain attack names ($\mathcal{R}^? \subseteq \mathbf{R}$).
- $\mathbf{src} : (\mathcal{R} \cup \mathcal{R}^?) \rightarrow (\mathcal{A} \cup \mathcal{A}^?)$ is the source function.
- $\mathbf{trg} : (\mathcal{R} \cup \mathcal{R}^?) \rightarrow (\mathcal{A} \cup \mathcal{A}^? \cup \mathcal{R} \cup \mathcal{R}^?)$ is the target function.

We also assume that $X \cap X^? = \emptyset$, for $X \in \{\mathcal{A}, \mathcal{R}\}$; and that for every $\alpha, \beta \in (\mathcal{R} \cup \mathcal{R}^?)$, $\alpha \neq \beta$ implies $\mathbf{v}(\alpha) \neq \mathbf{v}(\beta)$.

The previous definition is a neat combination of Definition 4 and Definition 6. It is conservative w.r.t. both definitions in the following sense. If $\mathbf{trg} : (\mathcal{R} \cup \mathcal{R}^?) \rightarrow (\mathcal{A} \cup \mathcal{A}^?)$, then \mathcal{IH} is reduced to an IAF (only first-order attacks). Analogously, if $\mathcal{A}^? = \mathcal{R}^? = \emptyset$, then we obtain an HOAF (without uncertainty).

Table 1 provides a recap of the types of (enriched) abstract argumentation frameworks which are covered by the IHOAF.

We are aware that one could also consider \mathbf{src} and/or \mathbf{trg} to be (partly) uncertain but this option faces at least two problems. Technically, there is no simple way of adapting the forthcoming notion of completion to uncertain source and target functions. Conceptually, it seems way less common/natural than our current choice. Consequently, this case is left for potential future work.

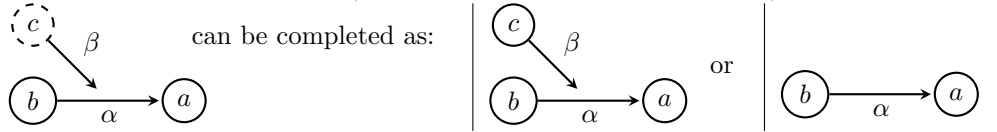
The next step is the definition of the notion of completion. But before giving a precise definition, we must analyse the impact of higher-order attacks on the notion of completion. Thus let us revisit the HOAF used in Example 3 and

Name	Def.	Sets	src :	trg :
AF	1	\mathcal{A}, \mathcal{R}	$\mathcal{R} \rightarrow \mathcal{A}$	$\mathcal{R} \rightarrow \mathcal{A}$
HOAF	6	\mathcal{A}, \mathcal{R}	$\mathcal{R} \rightarrow \mathcal{A}$	$\mathcal{R} \rightarrow (\mathcal{A} \cup \mathcal{R})$
IAF	4	$\mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^?$	$(\mathcal{R} \cup \mathcal{R}^?) \rightarrow (\mathcal{A} \cup \mathcal{A}^?)$	$(\mathcal{R} \cup \mathcal{R}^?) \rightarrow (\mathcal{A} \cup \mathcal{A}^?)$
IHOAF	9	$\mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^?$	$(\mathcal{R} \cup \mathcal{R}^?) \rightarrow (\mathcal{A} \cup \mathcal{A}^?)$	$(\mathcal{R} \cup \mathcal{R}^?) \rightarrow (\mathcal{A} \cup \mathcal{A}^? \cup \mathcal{R} \cup \mathcal{R}^?)$

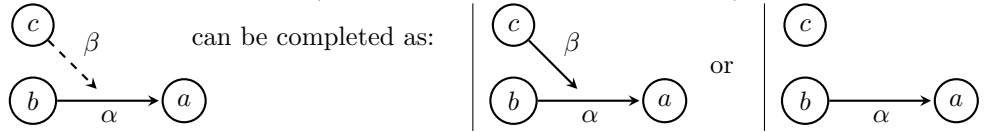
Table 1: The frameworks which are covered by an IHOAF $\mathcal{IH} = (\mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^?, \mathbf{src}, \mathbf{trg})$. For each framework, the sets which are not mentioned are empty.

imagine what should intuitively be the possible completions if each one of the elements is uncertain. Notice that the arguments of \mathcal{A} (resp. $\mathcal{A}^?$) are represented with plain (resp. dashed) nodes, the attacks of \mathcal{R} (resp. $\mathcal{R}^?$) with plain (resp. dashed) edges with the attack name as a label on the arrow.

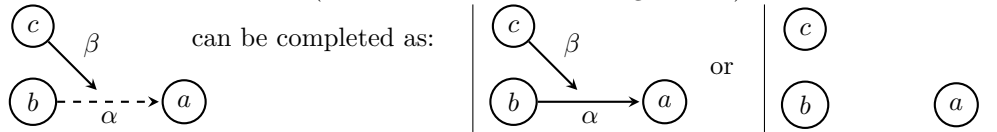
1. Either c is uncertain (all the other elements being certain):



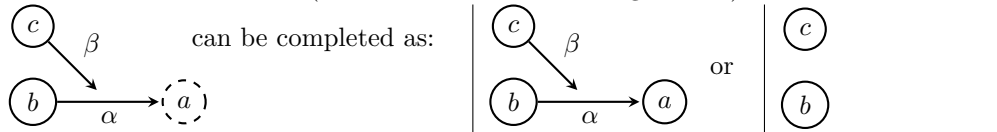
2. Or β is uncertain (all the other elements being certain):



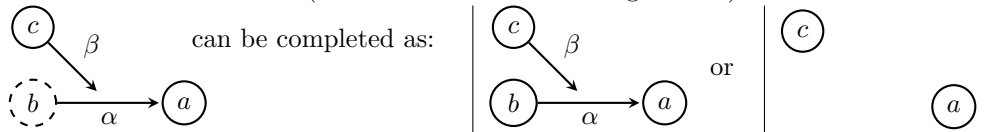
3. Or α is uncertain (all the other elements being certain):



4. Or a is uncertain (all the other elements being certain):



5. Or b is uncertain (all the other elements being certain):



The first interesting point is that, in the last two cases, we have a kind of “removal in cascade”: the removal of a (or b) implies the removal of the attack α from b to a and so the removal of the attack β targeting α . Hence, the

following question emerges: how to propagate the removal through the graph when computing the completions since, potentially, a given completion could be obtained by several chains of removal?

Another remark is that, here, only one kind of completion seems useful, because there is no semantic notion that should intuitively interact with the removal of uncertainty; unlike what happens with incomplete bipolar AFs [24].⁷

Back to Example 1, where c and β are both uncertain, it can be noticed that the three possible completions are the ones given in case 1 and case 2.

Following the previous discussion, the definition for completions is:

Definition 10 (IHOAF Completions) *Let $\mathcal{IH} = (\mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^?, \mathbf{src}, \mathbf{trg})$ be an IHOAF. A completion of \mathcal{IH} is any HOAF $\mathcal{H} = (\mathcal{A}_c, \mathcal{R}_c, \mathbf{src}_c, \mathbf{trg}_c)$ with:*

- $\mathcal{A} \subseteq \mathcal{A}_c \subseteq (\mathcal{A} \cup \mathcal{A}^?)$ and $\mathcal{R}_c \subseteq (\mathcal{R} \cup \mathcal{R}^?)$.
- $\mathbf{v}(\mathcal{R}) \cap (\mathcal{A}_c \times (\mathcal{A}_c \cup \mathcal{R}_c)) \subseteq \mathbf{v}(\mathcal{R}_c) \subseteq (\mathbf{v}(\mathcal{R} \cup \mathcal{R}^?) \cap (\mathcal{A}_c \times (\mathcal{A}_c \cup \mathcal{R}_c)))$.
- $\mathbf{src}_c = \mathbf{src} \cap (\mathcal{R}_c \times \mathcal{A}_c)$.
- $\mathbf{trg}_c = \mathbf{trg} \cap (\mathcal{R}_c \times (\mathcal{A}_c \cup \mathcal{R}_c))$.

We denote as $\text{completions}(\mathcal{IH})$ the set of all completions of \mathcal{IH} .

Note that the second constraint in Definition 10 can be equivalently replaced by:

1. $\forall \alpha \in \mathcal{R}, \alpha \in \mathcal{R}_c$ iff $\mathbf{src}(\alpha) \in \mathcal{A}_c$ and $\mathbf{trg}(\alpha) \in \mathcal{A}_c \cup \mathcal{R}_c$, and
2. $\forall \alpha \in \mathcal{R}^?, \alpha \in \mathcal{R}_c$ implies that $\mathbf{src}(\alpha) \in \mathcal{A}_c$ and $\mathbf{trg}(\alpha) \in \mathcal{A}_c \cup \mathcal{R}_c$.

The following proposition holds trivially, showing that our notion of completion is conservative w.r.t. the standard one (Definition 5):

Proposition 2 *Let $\mathcal{IH} = (\mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^?, \mathbf{src}, \mathbf{trg})$ be an IHOAF, if $\mathbf{trg} : \mathcal{R} \cup \mathcal{R}^? \rightarrow \mathcal{A} \cup \mathcal{A}^?$, then $\text{completions}(\mathcal{IH}) = \text{completions}(\mathcal{A}, \mathcal{A}^?, \mathbf{v}(\mathcal{R}), \mathbf{v}(\mathcal{R}^?))$.*

Using these completions, we can define semantics for IHOAFs:

Definition 11 (IHOAF structures) *Let $\mathcal{IH} = (\mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^?, \mathbf{src}, \mathbf{trg})$ be an IHOAF. We denote as $\sigma(\mathcal{IH})$ the set of all σ -structures of \mathcal{IH} under the semantics σ ($\sigma \in \{\text{pr}, \text{gr}, \text{co}, \text{st}\}$), and define it as $\sigma(\mathcal{IH}) = \{(S, \Gamma) \in 2^{(\mathcal{A} \cup \mathcal{A}^?) \times 2^{(\mathcal{R} \cup \mathcal{R}^?)}} \mid \exists \mathcal{H} \in \text{completions}(\mathcal{IH}) \text{ and } (S, \Gamma) \in \sigma(\mathcal{H})\}$.*

Example 5 *In the framework of Example 1 where both c and β are uncertain, each of the 3 possible completions has a unique preferred, grounded, stable and complete structure; thus, given $\sigma \in \{\text{co}, \text{pr}, \text{gr}, \text{st}\}$, $\sigma(\mathcal{IH}) = \{(\{a, b, c\}, \{\beta\}), (\{b\}, \{\alpha\}), (\{b, c\}, \{\alpha\})\}$. \square*

⁷In the case of IBAF, three types of completion can be defined (see [24]).

The following example illustrates the adequacy of the constraints assumed in the definition of completion.

Example 6 Let us consider the IHOAF \mathcal{IH} depicted in Figure 1 in which the arguments a and c certainly exist whereas the existence of the argument b is uncertain. Concerning the attacks, they are all certain except γ .

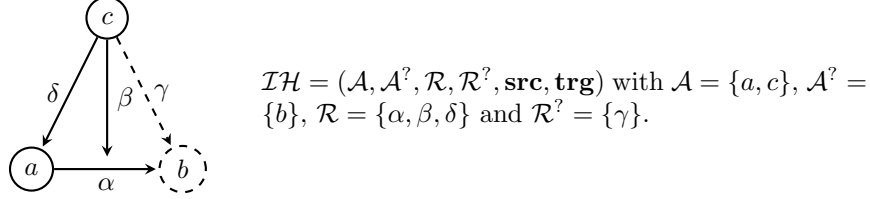


Figure 1: An Example of a IHOAF \mathcal{IH}

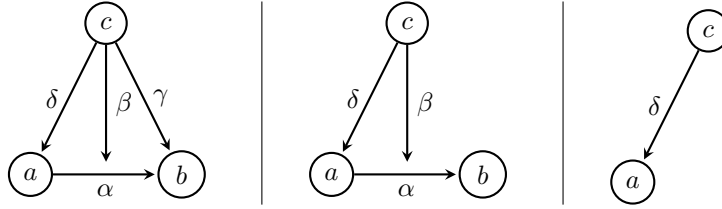


Figure 2: Three completions for the IHOAF \mathcal{IH}

Nevertheless, since the target of α is uncertain, α will exist only if its target also exists. Moreover, since α is the target of β the uncertainty of b also impacts the existence of β whereas it and its source and target are certain. So there are three possible completions given in Figure 2 and in one of them, β is missing because b is missing. This does not happen to δ , because a and c are not impacted by the uncertainty of b . \square

Some classical decision problems can be defined for IHOAF, i.e. possible credulous acceptability (PCA), necessary credulous acceptability (NCA) and their counterparts for skeptical acceptability (PSA and NSA):

Definition 12 Let $\mathcal{IH} = (\mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^?, \mathbf{src}, \mathbf{trg})$, $x \in (\mathcal{A} \cup \mathcal{R})$ and $\sigma \in \{\text{co}, \text{gr}, \text{pr}, \text{st}\}$. Four decision problems about the acceptability status of x can be defined:

- σ -PCA: $\exists \mathcal{H} \in \text{completions}(\mathcal{IH}), \exists U = (S, \Gamma) \in \sigma(\mathcal{H})$ such that $x \in S \cup \Gamma$?
- σ -NCA: $\forall \mathcal{H} \in \text{completions}(\mathcal{IH}), \exists U = (S, \Gamma) \in \sigma(\mathcal{H})$ such that $x \in S \cup \Gamma$?
- σ -PSA: $\exists \mathcal{H} \in \text{completions}(\mathcal{IH}), \forall U = (S, \Gamma) \in \sigma(\mathcal{H}), x \in S \cup \Gamma$?
- σ -NSA: $\forall \mathcal{H} \in \text{completions}(\mathcal{IH}), \forall U = (S, \Gamma) \in \sigma(\mathcal{H}), x \in S \cup \Gamma$?

Note that, in the definition of these problems, we only consider certain elements similarly to their initial formulation for IAFs [8]. Indeed, if the target element (argument or attack) is uncertain ($x \in \mathcal{A}^? \cup \mathcal{R}^?$), then the necessary variants of the decision problems are trivial (the answer is always “NO” since these elements are absent from some completions). Regarding the possible variants, it can be easily shown that x is possibly (credulously or skeptically) accepted in \mathcal{IH} if and only if it is also the case in \mathcal{IH}' , a modified version of \mathcal{IH} where x has been made certain. This is why formal definitions and computational studies of these problems are always limited to certain elements.

Example 7 *In the framework of Example 1 where both c and β are uncertain, a , c , α and β , which belong to every structure in some completions according to $\sigma \in \{\text{co, pr, gr, st}\}$, are thus σ -PSA (and σ -PCA), and b , which belongs to every structure in every completion, is σ -NSA (and σ -NCA). This argument b (“Sam told me that it will rain later”) is thus the only element accepted whatever be the interpretation of the framework; we may say that this is the only conclusion which can be firmly drawn from the discussion regarding the program of the day.*

Notice however that if in the framework of Example 1, it is b which is considered as uncertain (if Sam is known not to be reliable for instance) and c and β as certain, then the 2 possible completions, in this case, have a unique preferred, grounded, stable and complete structure: $\sigma(\mathcal{IH}) = \{(\{a, b, c\}, \{\beta\}), (\{a, c\}, \emptyset)\}$. Arguments a and c are then σ -NSA: going to the beach in the afternoon (argument a) can be concluded from the discussion. \square

Complexity of Reasoning with IHOAFs. We recall that RAFs can be polynomially transformed into AFs while preserving the semantics, for classical semantics $\sigma \in \{\text{co, pr, st, gr}\}$ [23, Proposition 32]. This implies that, under these semantics, all standard decision problems have the same complexity for HOAFs as for Dung’s AFs [19].

Let us investigate the complexity of reasoning with IHOAFs. We focus on the four acceptability problems given in Definition 12. An IHOAF without higher-order attacks (*i.e.* such that all attacks target an argument) is simply an IAF. This means that the complexity of reasoning with IAFs [8, 27] provides a lower bound to the complexity of reasoning with IHOAFs. Then we show that these lower bounds can be transformed into completeness results for each problem.

Proposition 3 *The complexity in the worst case for the acceptability problems defined in Definition 12 is given in the following table (the columns entitled “Cred. Acc” and “Skep. Acc” are only given for comparing with the RAF case, see [23]):*

σ	Cred. Acc (RAF)	σ -PCA	σ -NCA	Skep. Acc. (RAF)	σ -PSA	σ -NSA
gr	P	NP- c	coNP- c	P	NP- c	coNP- c
co	NP- c	NP- c	Π_2^P - c	P	NP- c	coNP- c
st	NP- c	NP- c	Π_2^P - c	coNP- c	Σ_2^P - c	coNP- c
pr	NP- c	NP- c	Π_2^P - c	Π_2^P - c	Σ_3^P - c	Π_2^P - c

Note that the existence of uncertainty significantly increases the complexity of the acceptability problems in the case of the grounded semantics (the credulous and skeptical acceptability problems for RAF are polynomial, see [23]) but the impact is less important for the other semantics (even if the complexity increases also for these semantics in the case of NCA and PSA because exponentially many completions may exist). The difference of complexity is the same between HOAFs and IHOAFs than between AFs and IAFs.

4 Logical Encoding of IHOAFs

We translate IHOAFs argument acceptability problems to the Dynamic Logic of Propositional Assignments (DL-PA) [3], a well-behaved variant of Propositional Dynamic Logic. This encoding extends previous works on the relation between abstract argumentation and DL-PA [16, 36, 24].

Note that, in this section, we have to constrain the studied framework to the use of *finite* universes since infinitary formulas are out of the scope of standard DL-PA. Hence, from now on, \mathbf{A} and \mathbf{R} are assumed to be finite.

Propositional variables. Let Prp be a denumerable set of propositional variables $\text{Prp} = \{p_1, p_2, \dots\}$. We suppose that Prp contains several kinds of distinguished variables capturing the statuses of arguments and relations between them.

- **argument variables**, $\text{ARG}_{\mathcal{A}} = \{\text{arg}_x \mid x \in \mathcal{A}\}$ for any $\mathcal{A} \subseteq \mathbf{A}$.
- **attack variables**, $\text{ATT}_{\mathcal{R}} = \{\text{r}_\alpha \mid \alpha \in \mathcal{R}\}$ for any $\mathcal{R} \subseteq \mathbf{R}$.
- **source variables**, $\text{SRC}_X = \{\text{src}_{\alpha,y} \mid (\alpha, y) \in X\}$ for any $X \subseteq (\mathbf{R} \times \mathbf{A})$.
- **target variables**, $\text{TRG}_X = \{\text{trg}_{\alpha,y} \mid (\alpha, y) \in X\}$ for any $X \subseteq (\mathbf{R} \times (\mathbf{A} \cup \mathbf{R}))$.
- **structure variables**, $\text{IN}_X = \{\text{in}_\alpha \mid \alpha \in X\}$ for any $X \subseteq (\mathbf{A} \cup \mathbf{R})$.

Let $\text{Prp}_{\mathbf{A} \cup \mathbf{R}} = \text{ARG}_{\mathbf{A}} \cup \text{IN}_{\mathbf{A} \cup \mathbf{R}} \cup \text{ATT}_{\mathbf{R}} \cup \text{SRC}_{\mathbf{R} \times \mathbf{A}} \cup \text{TRG}_{\mathbf{R} \times (\mathbf{A} \cup \mathbf{R})}$. We assume that $\text{Prp} \subseteq \text{Prp}_{\mathbf{A} \cup \mathbf{R}}$.

DL-PA grammar. The DL-PA language contains both formulas and programs, defined as follows:

$$\begin{array}{l} \text{For formulas: } \varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid [\pi]\varphi \quad (p \in \text{Prp}) \\ \text{For programs: } \pi ::= +p \mid -p \mid \varphi? \mid (\pi; \pi) \mid (\pi \cup \pi) \mid \pi^* \quad (p \in \text{Prp}) \end{array}$$

The intended meaning of formulas is as usual for atoms and the Boolean connectors. As for modal formulas, $[\pi]\varphi$ reads “ φ is true after every possible execution of π ”, so that the dual $\langle \pi \rangle \varphi$, defined as $\neg[\pi]\neg\varphi$, reads “there is a possible execution of π that makes φ true”. As for programs, their intended meaning is as follows: $+p$ (resp. $-p$) is the atomic program that makes p true (resp. false). $\varphi?$ is the program that tests whether φ is true. $(\pi; \pi')$ is the sequential

composition of π and π' (“first execute π and then π' ”). ($\pi \cup \pi'$) is the non-deterministic choice (“choose non-deterministically between π or π' and execute one of them”). Finally, π^* is the unbounded iteration of π (“execute π a finite number of times”).

Semantics of DL-PA. Given a propositional valuation $v \subseteq \text{Prp}$ (so v is the set of the true variables), **truth for formulas** φ and the **meaning of programs** $\|\pi\|$ is given by mutual recursion:

$$\left| \begin{array}{ll} v \models p & \text{if } p \in v \\ v \models [\pi]\varphi & \text{if } (v, v') \in \|\pi\| \text{ implies } v' \models \varphi \end{array} \right.$$

and as usual for the Boolean connectives; moreover, considering that, given a binary relation R , R^* denotes the reflexive and transitive closure of R , the interpretation of programs is:

$$\left| \begin{array}{ll} \|\!+p\| & = \{(v, v') \mid v' = v \cup \{p\}\} \\ \|\varphi?\| & = \{(v, v) \mid v \models \varphi\} \\ \|\pi \cup \pi'\| & = \|\pi\| \cup \|\pi'\| \end{array} \right| \left| \begin{array}{ll} \|\!-p\| & = \{(v, v') \mid v' = v \setminus \{p\}\} \\ \|\pi; \pi'\| & = \|\pi\| \circ \|\pi'\| \\ \|\pi^*\| & = \|\pi\|^* \end{array} \right.$$

Some useful DL-PA programs. Here are some useful abbreviations in our object language (where $P = \{p_1, \dots, p_n\}$ is a finite subset of Prp):

$$\left| \begin{array}{ll} \text{skip} & = \top? \\ \text{mkTrueOne}(P) & = \bigcup_{p \in P} (\neg p?; +p) = (\neg p_1?; +p_1) \cup \dots \cup (\neg p_n?; +p_n) \\ \text{mkFalseOne}(P) & = \bigcup_{p \in P} (p?; -p) = (p_1?; -p_1) \cup \dots \cup (p_n?; -p_n) \\ \text{mkTrueSome}(P) & = \dot{\bigcup}_{p \in P} (+p \cup \text{skip}) = (+p_1 \cup \text{skip}); \dots; (+p_n \cup \text{skip}) \\ \text{mkFalseAll}(P) & = \dot{\bigcup}_{p \in P} (-p) = -p_1; \dots; -p_n \\ \text{vary}(P) & = \dot{\bigcup}_{p \in P} (+p \cup -p) = (+p_1 \cup -p_1); \dots; (+p_n \cup -p_n) \\ \text{if } \varphi \text{ then } \pi \text{ else } \pi' & = (\varphi?; \pi) \cup (\neg\varphi?; \pi') \\ \text{if } \varphi \text{ then } \pi & = \text{if } \varphi \text{ then } \pi \text{ else skip} \\ \text{while } \varphi \text{ do } \pi & = (\varphi?; \pi)^*; \neg\varphi? \end{array} \right.$$

HOAFs and Structures as Propositional Valuations. From our hypothesis that Prp contains Prp_{AUR} , we can encode each HOAF $\mathcal{H} = (\mathcal{A}, \mathcal{R}, \text{src}, \text{trg})$ by its **associated propositional valuation** $v_{\mathcal{H}} = \text{ARG}_{\mathcal{A}} \cup \text{ATT}_{\mathcal{R}} \cup \text{SRC}_{\text{src}} \cup \text{TRG}_{\text{trg}}$.

Conversely, given $v \subseteq \text{Prp}_{\text{AUR}}$, we define the **quasi-HOAF associated to** v as $\mathcal{H}_v = (\mathcal{A}_v, \mathcal{R}_v, \text{src}_v, \text{trg}_v)$ where:

$$\left| \begin{array}{ll} \mathcal{A}_v & = \{x \mid \text{arg}_x \in v\}. \\ \mathcal{R}_v & = \{\alpha \mid \mathbf{r}_\alpha \in v\}. \\ \text{src}_v & = \{(\alpha, x) \mid \text{src}_{\alpha, x} \in v\} \cap (\mathcal{R}_v \times \mathcal{A}_v). \\ \text{trg}_v & = \{(\alpha, x) \mid \text{trg}_{\alpha, x} \in v\} \cap (\mathcal{R}_v \times (\mathcal{A}_v \cup \mathcal{R}_v)). \end{array} \right.$$

There are some valuations whose associated quasi-HOAFs are indeed HOAFs. These valuations could be identified syntactically by providing a formula that is true exactly at them. However, this is not needed since our reduction will depart from a given, arbitrary HOAF. This nuance was irrelevant in previous DL-PA encodings of argumentative structures [17, 36, 24], as one could easily force the associated quasi-AF to be an AF by definition (by restricting the

set of interactions). However, the more complex interactions in HOAFs make this strategy less appealing.

Given $v \subseteq \text{Prp}_{\mathbf{AUR}}$, the **quasi-structure associated to v** is defined as $U_v = (S_v, \Gamma_v)$ where $S_v = \{x \in \mathbf{A} \mid \text{in}_x \in v\}$ and $\Gamma_v = \{\alpha \in \mathbf{R} \mid \text{in}_\alpha \in v\}$. We can easily capture in our syntax that U_v is indeed a structure, and this will be useful later on: $\text{Structure} = \bigwedge_{x \in \mathbf{A}} (\text{in}_x \rightarrow \text{arg}_x) \wedge \bigwedge_{\alpha \in \mathbf{R}} (\text{in}_\alpha \rightarrow \text{r}_\alpha)$.

Computing Extensions of HOAFs in DL-PA. Let $x \in \mathbf{AUR}$, and define

$$\begin{aligned}
\text{ConflictFree} &= \text{Structure} \wedge \neg \bigvee_{\alpha \in \mathbf{R}} \left((\text{in}_\alpha \wedge \bigvee_{x \in \mathbf{A}} (\text{src}_{\alpha,x} \wedge \text{in}_x) \right. \\
&\quad \left. \wedge \bigvee_{y \in \mathbf{AUR}} (\text{trg}_{\alpha,y} \wedge \text{in}_y) \right) \\
\text{Acc}_x &= (\text{arg}_x \vee \text{r}_x) \wedge \bigwedge_{\alpha \in \mathbf{R}} \left((\text{r}_\alpha \wedge \text{trg}_{\alpha,x}) \rightarrow \right. \\
&\quad \left. \bigvee_{\beta \in \mathbf{R}} (\text{in}_\beta \wedge \bigvee_{y \in \mathbf{A}} (\text{src}_{\beta,y} \wedge \text{in}_y) \right. \\
&\quad \left. \wedge (\text{trg}_{\beta,\alpha} \vee \bigvee_{z \in \mathbf{A}} (\text{src}_{\alpha,z} \wedge \text{trg}_{\beta,\alpha})) \right) \\
\text{Complete} &= \text{ConflictFree} \wedge \bigwedge_{x \in \mathbf{AUR}} (\text{in}_x \leftrightarrow \text{Acc}_x) \\
\text{Preferred} &= \text{Complete} \\
&\quad \wedge \left[\text{mkTrueOne}(\text{IN}_{\mathbf{AUR}}); \text{mkTrueSome}(\text{IN}_{\mathbf{AUR}}) \right] \neg \text{Complete} \\
\text{Grounded} &= \text{Complete} \\
&\quad \wedge \left[\text{mkFalseOne}(\text{IN}_{\mathbf{AUR}}); \text{mkFalseSome}(\text{IN}_{\mathbf{AUR}}) \right] \neg \text{Complete} \\
\text{Defeated}_x &= \bigvee_{\alpha \in \mathbf{R}} \left(\text{in}_\alpha \wedge \bigvee_{y \in \mathbf{A}} (\text{src}_{\alpha,y} \wedge \text{in}_y \wedge \text{trg}_{\alpha,x}) \right) \\
\text{Stable} &= \text{ConflictFree} \wedge \bigwedge_{x \in \mathbf{AUR}} \left(((\text{arg}_x \vee \text{r}_x) \wedge \neg \text{in}_x) \rightarrow \text{Defeated}_x \right)
\end{aligned}$$

For $\sigma \in \{\text{co}, \text{pr}, \text{st}, \text{gr}\}$, we let φ_σ denote the corresponding formula in the previous list. For example, $\varphi_{\text{st}} = \text{Stable}$. Moreover, we define a guess-and-test simple DL-PA program to capture the computation of extensions:

$$\text{mkStru}^\sigma = \text{vary}(\text{IN}_{\mathbf{AUR}}); \varphi_\sigma?$$

Proposition 4 *Let \mathcal{H} be an HOAF, let $\sigma \in \{\text{co}, \text{pr}, \text{gr}, \text{st}\}$, let $v_1 \subseteq \text{Prp}_{\mathbf{AUR}} \setminus \text{IN}_{\mathbf{A}}$ such that $\mathcal{H}_{v_1} = \mathcal{H}$ then $\sigma(\mathcal{H}) = \{U_v \mid (v_1, v) \in \|\text{mkStru}^\sigma\|\}$.*

Example 8 *Recall the HOAF of Example 3, let us call it \mathcal{H}_1 . We have that $v_{\mathcal{H}_1} = \{\text{arg}_a, \text{arg}_b, \text{arg}_c, \text{r}_\alpha, \text{r}_\beta, \text{src}_{\alpha,b}, \text{src}_{\beta,c}, \text{trg}_{\alpha,a}, \text{trg}_{\beta,\alpha}\}$. Let us describe the execution of $\text{mkStru}^{\text{co}}$ at $v_{\mathcal{H}_1}$. First, we have that executing $\text{vary}(\text{IN}_{\mathbf{AUR}})$ at $v_{\mathcal{H}_1}$ leads us to all valuations of the form $v_{\mathcal{H}_1} \cup X$ where $X \subseteq \text{IN}_{\mathbf{AUR}}$ (32 different valuations!). However, from all these valuations, $v_2 = v_{\mathcal{H}_1} \cup \{\text{in}_a, \text{in}_b, \text{in}_c, \text{in}_\beta\}$ is the only one that “survives” to the test **Complete**?, hence, according to Proposition 4, $U_{v_1} = (\{a, b, c\}, \{\beta\})$ is the only complete structure of \mathcal{H}_1 . \square*

Computing Completions of IHOAFs in DL-PA. Programs used for previous encodings of incomplete argumentation frameworks [36, 24] do not straightforwardly work here. We need to be sure that the resulting valuation encodes a completion (see Definition 10 and the remark following it):

$$\begin{aligned}
\text{Constraint}_1 &= \bigwedge_{\alpha \in \mathcal{R}} \left(r_\alpha \leftrightarrow \left(\bigvee_{x \in \mathcal{A}} (\text{src}_{\alpha,x} \wedge \text{arg}_x) \wedge \bigvee_{y \in \mathcal{A} \cup \mathcal{R}} (\text{trg}_{\alpha,y} \wedge (\text{arg}_y \vee r_y)) \right) \right) \\
\text{Constraint}_2 &= \bigwedge_{\alpha \in \mathcal{R}^?} \left(r_\alpha \rightarrow \left(\bigvee_{x \in \mathcal{A}} (\text{src}_{\alpha,x} \wedge \text{arg}_x) \wedge \bigvee_{y \in \mathcal{A} \cup \mathcal{R}} (\text{trg}_{\alpha,y} \wedge (\text{arg}_y \vee r_y)) \right) \right) \\
\text{Constraint} &= \text{Constraint}_1 \wedge \text{Constraint}_2
\end{aligned}$$

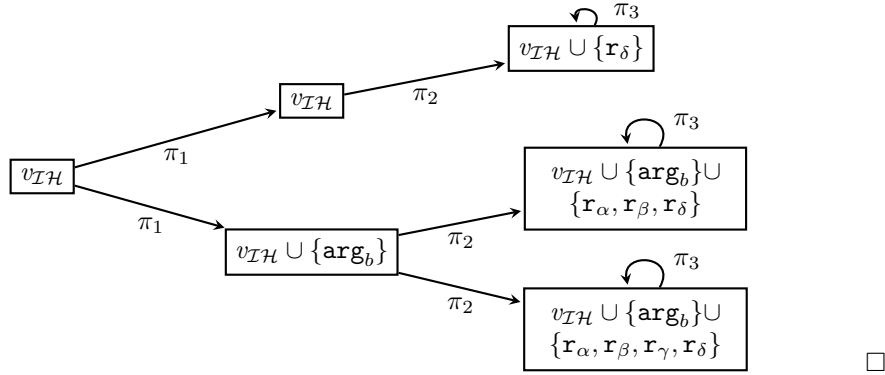
Let us define the **valuation associated to** $\mathcal{IH} = (\mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^?, \text{src}, \text{trg})$ as $v_{\mathcal{IH}} = \text{ARG}_{\mathcal{A}} \cup \text{SRC}_{\text{src}} \cup \text{TRG}_{\text{trg}}$. Completions of \mathcal{IH} are computed by executing the following program from $v_{\mathcal{IH}}$:

$$\text{mkComp}(\mathcal{IH}) = \text{mkTrueSome}(\text{ARG}_{\mathcal{A}^?}); \text{vary}(\text{ATT}_{\mathcal{R} \cup \mathcal{R}^?}); \text{Constraint}^?$$

Proposition 5 *Let $\mathcal{IH} = (\mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^?, \text{src}, \text{trg})$. Then:*

- *If $(v_{\mathcal{IH}}, v) \in \|\text{mkComp}(\mathcal{IH})\|$, then $(\mathcal{A}_v, \mathcal{R}_v, \text{src}_v, \text{trg}_v) \in \text{completions}(\mathcal{IH})$.*
- *If $(\mathcal{A}_c, \mathcal{R}_c, \text{src}_c, \text{trg}_c) \in \text{completions}(\mathcal{IH})$, then there is a $v \subseteq \text{Prp}_{\mathcal{A} \cup \mathcal{R}}$ such that $\mathcal{H}_v = (\mathcal{A}_c, \mathcal{R}_c, \text{src}_c, \text{trg}_c)$ and $(v_{\mathcal{IH}}, v) \in \|\text{mkComp}(\mathcal{IH})\|$.*

Example 9 *Recall the IHOAF of Example 6, \mathcal{IH} . We have that $v_{\mathcal{IH}} = \{\text{arg}_a, \text{arg}_c, \text{src}_{\alpha,a}, \text{src}_{\beta,c}, \text{src}_{\gamma,c}, \text{src}_{\delta,c}, \text{trg}_{\alpha,b}, \text{trg}_{\beta,\alpha}, \text{trg}_{\gamma,b}, \text{trg}_{\delta,a}\}$. The figure below represents the execution of $\text{mkComp}(\mathcal{IH})$, where $\pi_1 = \text{mkTrueSome}(\text{ARG}_{\mathcal{A}^?})$, $\pi_2 = \text{vary}(\text{ATT}_{\mathcal{R} \cup \mathcal{R}^?})$, and $\pi_3 = \text{Constraint}^?$. Note that many valuations related to $v_{\mathcal{IH}}$ and $v_{\mathcal{IH}} \cup \{\text{arg}_b\}$ through π_2 have been omitted for representational purposes since they do not pass the test performed by π_3 .*



Argument Acceptance in DL-PA We can finally formulate the desired reduction:

Proposition 6 *Given a $\mathcal{IH} = (\mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^?, \text{src}, \text{trg})$ and $x \in \mathcal{A} \cup \mathcal{R}$, then:*

- *the answer to σ -PCA is YES iff $v_{\mathcal{IH}} \models \langle \text{mkComp}(\mathcal{IH}); \text{mkStru}^\sigma \rangle \text{in}_x$.*
- *the answer to σ -NCA is YES iff $v_{\mathcal{IH}} \models [\text{mkComp}(\mathcal{IH})] \langle \text{mkStru}^\sigma \rangle \text{in}_x$.*

- the answer to σ -PSA is YES iff $v_{\mathcal{IH}} \models \langle \text{mkComp}(\mathcal{IH}) \rangle [\text{mkStru}^\sigma] \text{in}_x$.
- the answer to σ -NSA is YES iff $v_{\mathcal{IH}} \models [\text{mkComp}(\mathcal{IH}); \text{mkStru}^\sigma] \text{in}_x$.

5 Discussion and Conclusions

This paper is devoted to the study of IHOAFs: enriched frameworks for abstract argumentation taking into account higher-order attacks and considering at the same time that the elements (arguments or attacks) of this framework can be uncertain. Reasoning about IHOAFs is done through the notion of completion: any uncertain element can be considered present or absent and so several “variants” of the IHOAF may be built. Each variant is called a *completion* and corresponds to a classical HOAF without uncertainty. Then, the semantics of IHOAFs are defined by the application of the corresponding semantics on these completions. Complexity results are given for the possible/necessary credulous/skeptical acceptance problems under several semantics in IHOAFs. Finally, DL-PA programs for computing completions and extensions of IHOAFs are presented, and argument acceptance problems for IHOAFs are shown to be reducible to DL-PA model checking.

This work paves the way for a more general framework that would embed a higher-order support relation in addition to the higher-order attack relation: ongoing work on the combination of IBAFs [20, 24] and IHOAFs to obtain IHOBAFs (Incomplete Higher-Order Bipolar Argumentation Frameworks) is being conducted. Besides, since the HOAF semantics can be defined in several ways, it could be interesting to study the impact of the chosen definitions (for instance, RAF approach versus AFRA approach) from a more general point of view.

In terms of additional future works, several directions can be explored. Regarding the framework, a first idea would be to focus on the uncertainty aspect of IHOAFs by, e.g., considering a recursive form of uncertainty [34], or proposing a direct approach (i.e. without using completions) to define extension-based semantics in the style of [26]. Another idea could be the study of instantiations of IHOAFs into structured frameworks (see for instance [32, 37] for recent instantiations of IAFs). Dropping the uniqueness of the attack relation may also be investigated: there actually exist approaches such as [12], which considers that if the framework contains arguments and relations coming from different contexts or sources, then several attacks between the same two elements may occur. The consequences of these non-unique attacks on the definition of the completions, semantics and encoding may be studied.

Regarding computational aspects, an interesting point would be to implement the computation of semantics of the incomplete frameworks via the development of a DL-PA solver and to make some experiments for evaluating our encoding and comparing it with other approaches (for instance with a more direct computation of semantics, without using logics).

Two final research avenues can be mentioned: first, the question of explainabil-

ity for IHOAFs, notably to explain the possible/necessary credulous/skeptical acceptance of an element of the framework. A possible way to address this issue might be to start from the explainability approach of [1] for IAFs, and to extend it to IHOAFs. Another way would be to adapt explanations for AFs [14], in particular, visual approaches such as [15], to IHOAFs.

Second, the question of enforcement in IHOAFs may be addressed: how to make an element possibly/necessarily credulously/skeptically accepted by (minimally) changing the framework. In a situation such as the one of the running example Example 1, if one wishes that argument a be necessarily and not just possibly accepted, then changes on the certainty of arguments and attacks as indicated in Example 7 may be done. Enforcement in IHOAFs may be studied starting from the dynamic approaches for Control Argumentation Frameworks [25, 30] and IAFs [36].

To go even further, some approaches to explainability use enforcement results (they explain why an element is not acceptable by presenting the elements of the framework to be changed for it to be [14]): the last two research avenues may be combined in this sense.

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Appendix

Proofs for Section 2

Proposition 1 Let \mathcal{H} be an HOAF. The preferred structures of \mathcal{H} are its \subseteq -maximal complete structures.

Proof: Let $U = (S, \Gamma)$ be a preferred structure of \mathcal{H} . From [12], U is a complete structure as well. Assume it is not a \subseteq -maximal complete structure. Then, there is $U' = (S', \Gamma')$ a complete structure s.t. $U \subset U'$. However, by definition, U' is admissible (i.e. it is conflict-free and $\text{Acc}(U) \subseteq S' \cup \Gamma'$), which contradicts the fact that U is preferred. So U is a \subseteq -maximal complete structure.

Then, let $U = (S, \Gamma)$ be a \subseteq -maximal complete structure. By definition, it is admissible. If U is not preferred, it means that there is an admissible structure $U' = (S', \Gamma')$ s.t. $U \subset U'$. However, we know that such a preferred structure is complete [12], so this means that U is not a \subseteq -maximal complete structure, hence a contradiction. So U is preferred. \square

Proofs for Section 3

Proposition 3. The complexity in the worst case for the acceptability problems given in Definition 12 is:

1. For $\sigma \in \{\text{gr}, \text{st}, \text{co}, \text{pr}\}$, σ -PCA is NP-complete.
2. For $\sigma \in \{\text{gr}, \text{st}, \text{co}\}$, σ -NSA is coNP-complete and it is Π_2^P -complete for $\sigma = \text{pr}$.
3. For $\sigma \in \{\text{st}, \text{co}, \text{pr}\}$, σ -NCA is Π_2^P -complete, and it is coNP-complete for $\sigma = \text{gr}$.
4. For $\sigma \in \{\text{co}, \text{gr}\}$, σ -PSA is NP-complete, it is Σ_2^P -complete for $\sigma = \text{st}$, and Σ_3^P -complete for $\sigma = \text{pr}$.

Proof: Let us consider each item:

1. σ -PCA is NP-complete for all the considered semantics with IAFs [8]. So it is NP-hard for IHOAFs. Now let us prove the NP-membership. Guess a completion \mathcal{H} of \mathcal{IH} and a structure $U = (S, \Gamma)$ such that $x \in S$. For all semantics (except $\sigma = \text{pr}$), verifying whether U is a σ -structure is polynomial. For the case of preferred semantics, verifying whether U is complete is sufficient to prove that x is possibly credulously accepted: indeed, since pr -structures are \subseteq -maximal complete structures, credulous acceptability under these semantics coincides. So, we exhibit a NP algorithm for solving σ -PCA, hence the result.
2. Again, lower bounds are provided by the known complexity of this problem for IAFs [8]. For the upper bound, let us consider this non-deterministic algorithm. Guess a completion \mathcal{H} of \mathcal{IH} and a

structure $U = (S, \Gamma)$ such that $x \notin S \cup \Gamma$. Verifying whether U is a σ -structure is polynomial for $\sigma \in \{\text{gr}, \text{st}, \text{co}\}$, and **coNP**-complete for $\sigma = \text{pr}$. This induces a **NP** (respectively Σ_2^P) algorithm for the complement problem of σ -NSA, hence the result.

3. Again the lower bounds come from the known complexity of reasoning with IAFs [8]. For the upper bound, we can non-deterministically guess a completion \mathcal{H} of \mathcal{IH} , and check (in **NP** for all semantics except $\sigma = \text{gr}$, in which case it is polynomial) whether x is not credulously accepted. So we obtain a Σ_2^P (respectively **NP**) algorithm for the complement problem of σ -NCA, and we deduce the result.
4. As in the previous cases, we deduce the lower bounds from the known complexity of σ -PSA with IAFs [8]. For the upper bound, we can solve the problem by guessing a completion \mathcal{H} of \mathcal{IH} , and then checking (in polynomial time for $\sigma \in \{\text{co}, \text{gr}\}$, in **coNP** for $\sigma = \text{st}$ and in Π_2^P for $\sigma = \text{pr}$) whether x is skeptically accepted. This leads us to the result.

□

Proofs for Section 4

Proposition 4 Let \mathcal{H} be an HOAF, let $\sigma \in \{\text{co}, \text{pr}, \text{gr}, \text{st}\}$, let $v_1 \subseteq \text{Prp}_{\text{AUR}} \setminus \text{IN}_{\text{AUR}}$ such that $\mathcal{H}_{v_1} = \mathcal{H}$ then: $\sigma(\mathcal{H}) = \{U_v \mid (v_1, v) \in \|\text{mkStru}^\sigma\|\}$.

Proof:[sketched] The claim follows from the meaning of mkStru^σ and the following lemma, whose proof poses no difficulty.

Lemma 1 Let \mathcal{H} be an HOAF, let $\sigma \in \{\text{co}, \text{pr}, \text{gr}, \text{st}\}$, let $v_1 \subseteq \text{Prp}_{\text{AUR}} \setminus \text{IN}_{\text{AUR}}$ such that $\mathcal{H}_{v_1} = \mathcal{H}$, let $v \subseteq \text{IN}_{\text{AUR}}$, let $x \in \mathcal{A} \cup \mathcal{R}$, then:

- $v_1 \cup v \models \text{Structure}$ iff U_v is a structure of \mathcal{H} .
- $v_1 \cup v \models \text{ConflictFree}$ iff U_v is conflict-free w.r.t. \mathcal{H} .
- $v_1 \cup v \models \text{Structure} \wedge \text{Acc}_x$ iff x is acceptable w.r.t. U_v .
- $v_1 \cup v \models \text{Structure} \wedge \text{Defeated}_x$ iff x is defeated by U_v .
- $v_1 \cup v \models \varphi_\sigma$ iff $U_v \in \sigma(\mathcal{H})$.⁸

□

Proposition 5 Let $\mathcal{IH} = (\mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^?, \text{src}, \text{trg})$. Then:

- If $(v_{\mathcal{IH}}, v) \in \|\text{mkComp}(\mathcal{IH})\|$, then $(\mathcal{A}_v, \mathcal{R}_v, \text{src}_v, \text{trg}_v) \in \text{completions}(\mathcal{IH})$.
- If $(\mathcal{A}_c, \mathcal{R}_c, \text{src}_c, \text{trg}_c) \in \text{completions}(\mathcal{IH})$, then there is a $v \subseteq \text{Prp}_{\text{AUR}}$ such that $\mathcal{H}_v = (\mathcal{A}_c, \mathcal{R}_c, \text{src}_c, \text{trg}_c)$ and $(v_{\mathcal{IH}}, v) \in \|\text{mkComp}(\mathcal{IH})\|$.

⁸For $\sigma = \text{pr}$, we need Proposition 1.

Proof: For the first item, suppose that

$(v_{\mathcal{I}\mathcal{H}}, v) \in \|\text{mkComp}(\mathcal{I}\mathcal{H})\|$, which, by definition of mkComp amounts to $(v_{\mathcal{I}\mathcal{H}}, v) \in \|\text{mkTrueSome}(\text{ARG}_{\mathcal{A}^?}; \text{vary}(\text{ATT}_{\mathcal{R} \cup \mathcal{R}^?}); \text{Constraint}^?)\|$.

We want to show that $(\mathcal{A}_v, \mathcal{R}_v, \text{src}_v, \text{trg}_v) \in \text{completions}(\mathcal{I}\mathcal{H})$, which amounts, by definition of completion (Definition 10) to showing four things:

- (1) $\mathcal{A} \subseteq \mathcal{A}_v \subseteq \mathcal{A} \cup \mathcal{A}^?$ and $\mathcal{R}_v \subseteq \mathcal{R} \cup \mathcal{R}^?$.
- (2.1) $\forall \alpha \in \mathcal{R}, \alpha \in \mathcal{R}_v$ iff $\text{src}(\alpha) \in \mathcal{A}_v$ and $\text{trg}(\alpha) \in \mathcal{A}_v \cup \mathcal{R}_v$.
- (2.2) $\forall \alpha \in \mathcal{R}^?, \alpha \in \mathcal{R}_v$ implies that $\text{src}(\alpha) \in \mathcal{A}_v$ and $\text{trg}(\alpha) \in \mathcal{A}_v \cup \mathcal{R}_v$.
- (3) $\text{src}_v = \text{src} \cap (\mathcal{R}_v \times \mathcal{A}_v)$.
- (4) $\text{trg}_v = \text{trg} \cap (\mathcal{R}_v \times (\mathcal{A}_v \cup \mathcal{R}_v))$.

For the first inclusion of (1), note that $\mathcal{A} \subseteq \mathcal{A}_v$, because $\mathcal{A} = \mathcal{A}_{v_{\mathcal{I}\mathcal{H}}}$ by definition and because mkComp does not make false any variable from $\text{ARG}_{\mathcal{A}}$. Moreover, $\mathcal{A}_v \subseteq \mathcal{A} \cup \mathcal{A}^?$ holds because argument variables that hold at v but do not hold at $v_{\mathcal{I}\mathcal{H}}$ can only belong to $\text{ARG}_{\mathcal{A}^?}$, by the definition of mkComp . Finally, for showing that $\mathcal{R}_v \subseteq \mathcal{R} \cup \mathcal{R}^?$, it is enough to see that no attack variable is true at $v_{\mathcal{I}\mathcal{H}}$ and that the execution of mkComp might only make true attack variables from $\text{ATT}_{\mathcal{R} \cup \mathcal{R}^?}$.

As for (2), note that $v \models \text{Constaint}$ (because of the test $\text{Constraint}^?$ in the definition of mkComp). Recall that $\text{Constaint} = \text{Constraint}_1 \wedge \text{Constraint}_2$. It is a matter of simple propositional reasoning to verify that $v \models \text{Constraint}_1$ (resp. $v \models \text{Constraint}_2$) guarantee that satisfaction of (2.1) (resp. (2.2)).

Showing that (3) and (4) are true can be done through very similar arguments. Let us develop the one for (3). First, note that $\{(\alpha, x) \mid \text{src}_{\alpha, x} \in v\} = \{(\alpha, x) \mid \text{src}_{\alpha, x} \in v_{\mathcal{I}\mathcal{H}}\}$, because the execution of mkComp does not alter the value of source variables. Hence, we have that $\text{src}_v = \{(\alpha, x) \mid \text{src}_{\alpha, x} \in v_{\mathcal{I}\mathcal{H}}\} \cap (\mathcal{R}_v \times \mathcal{A}_v)$, by definition of src_v . But we also know that $\{(\alpha, x) \mid \text{src}_{\alpha, x} \in v_{\mathcal{I}\mathcal{H}}\} = \text{src}$, by definition of $v_{\mathcal{I}\mathcal{H}}$. Hence we have that $\text{src}_v = \text{src} \cap (\mathcal{R}_v \times \mathcal{A}_v)$.

For the second item, suppose that $(\mathcal{A}_c, \mathcal{R}_c, \text{src}_c, \text{trg}_c) \in \text{completions}(\mathcal{I}\mathcal{H})$, which is equivalent, by definition of completion, to the conjunction of the following assertions:

- (1) $\mathcal{A} \subseteq \mathcal{A}_c \subseteq \mathcal{A} \cup \mathcal{A}^?$ and $\mathcal{R}_c \subseteq (\mathcal{R} \cup \mathcal{R}^?)$.
- (2.1) $\forall \alpha \in \mathcal{R}, \alpha \in \mathcal{R}_c$ iff $\text{src}(\alpha) \in \mathcal{A}_c$ and $\text{trg}(\alpha) \in \mathcal{A}_c \cup \mathcal{R}_c$.
- (2.2) $\forall \alpha \in \mathcal{R}^?, \alpha \in \mathcal{R}_c$ implies that $\text{src}(\alpha) \in \mathcal{A}_c$ and $\text{trg}(\alpha) \in \mathcal{A}_c \cup \mathcal{R}_c$.
- (3) $\text{src}_c = \text{src} \cap (\mathcal{R}_c \times \mathcal{A}_c)$.
- (4) $\text{trg}_c = \text{trg} \cap (\mathcal{R}_c \times (\mathcal{A}_c \cup \mathcal{R}_c))$.

We define the following propositional valuation, that will serve as a witness of the existential claim we want to prove: $v = \text{ARG}_{\mathcal{A}_c} \cup \text{ATT}_{\mathcal{R}_c} \cup \text{SRC}_{\text{src}_c} \cup \text{TRG}_{\text{trg}_c}$. Moreover, we define the following auxiliary valuation: $v_1 = v_{\mathcal{H}} \cup \{\text{arg}_x \mid x \in \mathcal{A}_c\}$. Note that $v = v_1 \cup \{\text{r}_\alpha \mid \alpha \in \mathcal{R}_c\}$. Now, we need to show that the following hold:

Lemma 2

- (a) $(v_{\mathcal{H}}, v_1) \in \|\text{mkTrueSome}(\text{ARG}_{\mathcal{A}^?})\|$.
- (b) $(v_1, v) \in \|\text{vary}(\text{ATT}_{\mathcal{R} \cup \mathcal{R}^?})\|$.
- (c) $v \models \text{Constraint}$.
- (d) $\mathcal{H}_v = (\mathcal{A}_c, \mathcal{R}_c, \text{src}_c, \text{trg}_c)$.

Items (a) and (b) follow easily from (1). (c) follows from (2.1) and (2.2). As for (d), it can be deduced from (3) and (4). Let us sketch the proof of the last statement. Note that $\mathcal{A}_v = \mathcal{A}_c$ and $\mathcal{R}_v = \mathcal{R}_c$ by definition of v . For showing that $\text{src}_v = \text{src}_c$, we first have that $\text{src}_v = \{(\alpha, x) \mid \text{src}_{\alpha, x} \in v_{\mathcal{I}\mathcal{H}}\} \cap (\mathcal{R}_v \times \mathcal{A}_v)$, because mkComp does not alter the value of source variables. Moreover, $\{(\alpha, x) \mid \text{src}_{\alpha, x} \in v_{\mathcal{I}\mathcal{H}}\} \cap (\mathcal{R}_v \times \mathcal{A}_v) = \{(\alpha, x) \mid \text{src}_{\alpha, x} \in v_{\mathcal{I}\mathcal{H}}\} \cap (\mathcal{R}_c \times \mathcal{A}_c)$ (because $\mathcal{A}_v = \mathcal{A}_c$ and $\mathcal{R}_v = \mathcal{R}_c$, as we have already argued). We also have that $\{(\alpha, x) \mid \text{src}_{\alpha, x} \in v_{\mathcal{I}\mathcal{H}}\} \cap (\mathcal{R}_c \times \mathcal{A}_c) = \text{src} \cap (\mathcal{R}_c \times \mathcal{A}_c)$ (by definition of $v_{\mathcal{I}\mathcal{H}}$), which is the same set that src_c , by (3). Showing that $\text{trg}_v = \text{trg}_c$ follows a similar line of reasoning, using (4) instead of (3).

Finally, note that (a), (b) and (c) jointly imply that $(v_{\mathcal{H}}, v) \in \|\text{mkComp}(\mathcal{I}\mathcal{H})\|$, by definition of mkComp and the meaning of the DL-PA operator \cdot . \square

Proposition 6 Given a $\mathcal{I}\mathcal{H} = (\mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^?, \text{src}, \text{trg})$ and $x \in \mathcal{A} \cup \mathcal{R}$, then:

- the answer to σ -PCA is YES iff $v_{\mathcal{I}\mathcal{H}} \models \langle \text{mkComp}(\mathcal{I}\mathcal{H}); \text{mkStru}^\sigma \rangle \text{in}_x$.
- the answer to σ -NCA is YES iff $v_{\mathcal{I}\mathcal{H}} \models [\text{mkComp}(\mathcal{I}\mathcal{H})] \langle \text{mkStru}^\sigma \rangle \text{in}_x$.
- the answer to σ -PSA is YES iff $v_{\mathcal{I}\mathcal{H}} \models \langle \text{mkComp}(\mathcal{I}\mathcal{H}) \rangle [\text{mkStru}^\sigma] \text{in}_x$.
- the answer to σ -NSA is YES iff $v_{\mathcal{I}\mathcal{H}} \models [\text{mkComp}(\mathcal{I}\mathcal{H}); \text{mkStru}^\sigma] \text{in}_x$.

Proof: We prove the first item. The rest of the proofs share the same structure.

From left to right, suppose that the answer to σ -PCA is YES. This implies, by definition, that $\exists \mathcal{H} \in \text{completions}(\mathcal{I}\mathcal{H}), \exists U = (S, \Gamma) \in \sigma(\mathcal{H})$ such that $x \in S \cup \Gamma$. Let \mathcal{H}_c be such a HOAF. The second bullet of Proposition 5 implies that there is a $v_1 \subseteq \text{Prp}_{\mathcal{A} \cup \mathcal{R}}$ such that $\mathcal{H}_c = \mathcal{H}_{v_1}$ and $(v_{\mathcal{I}\mathcal{H}}, v_1) \in \|\text{mkComp}(\mathcal{I}\mathcal{H})\|$. The latter implies, using Proposition 4, that there is a $v_2 \subseteq \text{Prp}_{\mathcal{A} \cup \mathcal{R}}$ such that $(v_1, v_2) \in \|\text{mkStru}^\sigma\|$ and $x \in S_{v_2} \cup \Gamma_{v_2}$. Finally the latter implies, by the meaning of $\langle \cdot \rangle$ and \cdot ; that $v_{\mathcal{I}\mathcal{H}} \models \langle \text{mkComp}(\mathcal{I}\mathcal{H}); \text{mkStru}^\sigma \rangle \text{in}_x$.

From right to left, suppose that $v_{\mathcal{I}\mathcal{H}} \models \langle \text{mkComp}(\mathcal{I}\mathcal{H}); \text{mkStru}^\sigma \rangle \text{in}_x$. The latter is equivalent, by the meaning of $\langle \cdot \rangle$ and \cdot ; to claiming that there are v_1 and v_2 such that $(v_{\mathcal{H}}, v_1) \in \|\text{mkComp}(\mathcal{I}\mathcal{H})\|$, $(v_1, v_2) \in \|\text{mkStru}^\sigma\|$ and $\text{in}_x \in v_2$. By the first item of Proposition 5, we have that $(\mathcal{A}_{v_1}, \mathcal{R}_{v_1}, \text{src}_{v_1}, \text{trg}_{v_1}) \in \text{completions}(\mathcal{I}\mathcal{H})$. By Proposition 4, we have that $(S_{v_2}, \Gamma_{v_2}) \in \sigma((\mathcal{A}_{v_1}, \mathcal{R}_{v_1}, \text{src}_{v_1}, \text{trg}_{v_1}))$. Since we knew that $\text{in}_x \in v_2$, which is equivalent to $x \in S_{v_2} \cup \Gamma_{v_2}$, the latter two statements imply that the answer to σ -PCA (with input \mathcal{H} and x) is YES. \square