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Numerical schemes for the solution of the damped wave equation *

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Abstract

The present work deals with the numerical resolution of the damped wave equation. Thus, we present various numerical explicit and implicit schemes for problems defined in one-, two- and three-dimensional domain. In the case of one-dimensional problem, we consider, on one hand, Dirichlet boundary value conditions and in the other hand, mixed boundary value conditions. In the two- and three-dimensional cases, we consider only the case of Dirichlet boundary value conditions. However, in all cases, we analyze the stability and the truncation error of the presented schemes. Concerning the implicit schemes, we survey the main classical numerical linear algebra algorithms and show that they can be applied successfully to the numerical solution of the target problem, in the two- and three-dimensional cases on sequential or multiprocessor computers. We briefly consider also the cases where, on the one hand, the coefficients of the problem are not constant and, on the other hand, the case where the linear problem is perturbed by a singlevalued or a multivalued diagonal operator and we explain how to solve implicitly this type of problem on a multiprocessor. This study is completed by numerical simulations in the case of one-dimensional problems. Finally, to study the stability of the schemes, we can either use the classical Von Neumann method or the matricial one when the discretization matrix in space is normal. In this case, it is necessary to have the eigenvalues of the space discretization matrices. Thus, an appendix presents the determination of these eigenvalues in academic situations.

*Dedicated to Michela Redivo-Zaglia for her birthday and retirement and for the 65th-birthday of Hassane Sadok.

Keywords: Damped wave equation, telegrapher equation, stability, truncation error, linear systems, parallel solution.

1 Introduction.

The damped wave equation can be used to model many phenomena in electrical engineering, mechanical engineering and other fields. This is a second-order hyperbolic equation in time, which in addition to the second-order derivation terms in space and time present in the wave equation formulation, includes an additional lower-order derivation term with respect to time. Note that this equation can be classically reduced to a system of two first-order hyperbolic equations in time, each of which constitutes a transport or advection equation.

Generally speaking, the propagation of waves of any kind is one of the simplest and most common phenomena with which scientists are confronted. From everyday life (propagation of sounds, vibrations, waves, radar waves or waves produced in telecommunications) to the scale of the universe (due to the propagation effects of electromagnetic waves or gravity) and to that of the atom (due to spontaneous or simulated emissions and interference between particles) it is the emission and reception of waves that constitutes our privileged means of knowledge of the world around us.

In electrical engineering, this type of equation is known as the telegrapher equation. This last equation is derived from Maxwell equations when studying the propagation of electromagnetic waves in conductors or transmission lines; in particular, it takes into account the effects of the line resistance (R), inductance (L), conductance (G) and capacitance (C). In the framework of Maxwell equations, when we simplify the problem to a one-dimensional geometry with a linear conductor, we obtain two types of telegrapher equations for voltage V and current I :

$$\frac{\partial^2 V}{\partial x^2} = LC \frac{\partial^2 V}{\partial x^2} + (RC + LG) \frac{\partial V}{\partial t}, \quad (1)$$

$$\frac{\partial^2 I}{\partial x^2} = LC \frac{\partial^2 I}{\partial x^2} + (RC + LG) \frac{\partial I}{\partial t}. \quad (2)$$

These equations show how voltage or current disturbances propagate along lines.

This telegrapher equation can be related to Maxwell equations. Indeed, the latter are derived from :

- Faraday law : $\nabla \times E \equiv \text{curl}(E) = -\frac{\partial B}{\partial t}$,

- Maxwell-Ampère law, taking into account displacement currents : $\nabla \times H \equiv \text{curl}(H) = J + \frac{\partial D}{\partial t}$,

- Gauss law for the electric field : $\nabla \cdot D \equiv \text{grad}(D) = \rho$,

- Gauss law for the magnetic field: $\nabla \cdot B \equiv \text{grad}(B) = 0$,

where E denotes the electric field, B is the magnetic induction, H is the magnetic field, J is the current density of the source, D is the electric displacement, ρ is the volume charge density.

By choosing a simplified model involving propagation in a single direction and combining Maxwell equations and the constitutive $J = \sigma \cdot E$, where σ represents the surface charge density, we can classically obtain relations similar to those of the telegrapher equations. For example, in a conductor, inductance L and resistance R are associated with the magnetic and

conductive properties of the material, while capacitance C and conductance G are linked to their dielectric properties. The following applications can be listed:

- **Transmission lines (coaxial lines or cables):** a conventional transmission line modeled by the telegrapher equations can be analyzed as a distributed system of inductances and capacitances. In this case, the simplified Maxwell equations give the telegrapher equations directly. This model is used in the study of coaxial cables, two-wire lines, etc., to describe how electrical signals propagate and dissipate,
- **Waveguides:** when considering waveguides, by neglecting certain components (such as the longitudinal field) the telegrapher equation can be seen as an approximation of transverse electromagnetic wave propagation in guiding structures,
- **Long antennas and radio wave propagation:** for antennas or long conductors subjected to radio waves, the telegrapher equation can be used to model signal propagation along the conductor with losses and capacitive effects.

Note that when the coefficient $d = (RC + LG)$ is strictly positive, and when a coefficient a associated with the value of V or I is zero, or also if $d > a > 0$, the above equations represent the damped wave equation and the telegrapher equation. In fact, the positive coefficient d represents a deceleration force proportional to the speed and models a damping term. Thus, this term takes into account additional physical phenomena.

For equations (1) and (2) we can also use a system of two first-order hyperbolic equations, one describing the evolution of an electric potential and the other that of the current on a power line, these physical quantities being functions of space and time. This new model is obviously applicable to the description of the behavior of any line, and takes into account transmission and reflection phenomena on a transmission line, whether it is used for telegraph, telephone or any other purpose, as well as power grid distribution lines (see [1] and [2]). However, due to complex analytical computations, note that the numerical analysis of the resulting numerical scheme required to the numerical resolution of the damped wave equation is not very easy.

In signal processing, the aim is often to reconstruct a degraded input signal by analyzing the output signal. This type of subject is classically addressed by the control of a distributed parameter system. Indeed, in certain situations, thanks to the use of a power line, the knowledge of the precise physical laws of transmission and the continuous data dependencies must be taken into account. We correct the distortion of the signal along the power line, modeled by the telegrapher equation, which amounts to modify the input signal to correct the distortion of the output signal (see [3]).

Advances in brain observation techniques such as Functional Magnetic Resonance Imaging (or Electroencephalogram) have not enabled neuroscientists to obtain high spatial and temporal resolution of the electrical activity of neurons in the brain. As a result, a model based on the damped wave equation proposed by Jirsa and Haken in 1996 was able to correctly describe the function of brain activity at each point and at any time (see [4]).

In mechanics, the coefficient of the first derivative in time present in the damped wave equation represents the resistance of the ambient medium, typically the air in the case of a vibrating string, which produces energy dissipation. This may be due to the fact that the friction of solids in the fluid induces forces opposed to their displacement, the intensity of which is proportional to their speed.

The telegraph equation is also involved in the case of isotropic small angle scattering from the Boltzmann transport equation for charged particles for the control of thermonuclear fusion

schemes and in biological effects of radiation. Charged particles transport also plays a fundamental role in magnetic fusion scheme. The interaction mechanism of most concern in the transport of energetic charged particles in plasma involves small angle scattering collisions.

Except in particular situations, the damped wave equation is difficult to solve analytically, and numerical solution are provided. However, to the best of our knowledge, there are few contributions in the literature on the numerical solution of this equation. The aim of the present study is therefore first to propose a catalog of numerical schemes for solving the one-dimensional damped wave equation with both Dirichlet and mixed boundary conditions. From these one-dimensional numerical schemes, we extend this study to the establishment of numerical schemes for two-dimensional and three-dimensional problems only in the case of the damped wave equation equipped with Dirichlet boundary conditions; in fact, in two-dimensional or three-dimensional situations, taking into account other types of boundary conditions of the mixed type on the part of the boundary is complex to consider, for the numerical analysis of the schemes in order to be implemented. Indeed, even in the case of simple geometry of the domain Ω where the equation is defined, the condition concerning the normal derivative can occur on only one or several faces of the domain Ω , which would make the presentation of the present study considerably cumbersome; in this case, the numerical analysis of the schemes must be carried out by using numerical techniques.

In the present study, we consider first explicit numerical schemes, which allow us to calculate the solution provided that a stability condition of the Courant-Friedrich-Levy type (C.F.L.) is satisfied. The C.F.L. condition is too restrictive, as it requires very small time discretization steps, which leads to very costly computations. Indeed, to ensure stability, in the case where the diffusion coefficient is equal to one, we are forced to take time discretization step values of the same order as those of the space discretization step. Thus implicit numerical schemes are also considered and shown to be unconditionally stable, allowing relatively larger time steps to be chosen than those used when using explicit numerical schemes. However the use of implicit numerical schemes requires the resolution of large sparse linear systems, the matrices defining these systems having interesting properties, i.e. symmetry, definite positivity and strict diagonal dominance, which guarantees the invertibility of these matrices, in addition to sparsity properties. Thus, to complete the presented study, we recall a number of classical numerical algorithms that can be used to solve large dimensional sparse linear systems. For both explicit and implicit numerical schemes, this paper is completed by a study of the truncation error occurring during discretization in space and time, and by an evaluation of the energy of the system, this property establishing a continuous dependence on the data of the problem (i.e. the initial conditions, the second member of the problem and the time interval in which it is defined). Hence, by admitting the existence of the solution of the problem, the evaluation of the energy classically shows by absurdity that the solution is unique (see [5]).

The present manuscript is divided into six sections. Sections 2 and 3 allow the establishment of numerical schemes on the one hand for the one-dimensional problem equipped with Dirichlet boundary conditions and on the other hand for the one equipped with mixed type boundary conditions, this latter case not being treated in the literature. In these two sections, discretization schemes are also shown when the coefficient of the Laplacian is not constant but also depends on the space and time variables. This type of approximation is equally valid for two-dimensional and three-dimensional problems. Then section 4 extends the numerical schemes established in section 2 to two-dimensional and three-dimensional

problems, and when implicit numerical schemes are used, provides briefly an overview of the classical algorithms that can be used to implement numerical schemes, such as sequential relaxation algorithms, alternating directions method, which has the advantage of obtaining low matrix bandwidths, allowing the use of efficient algorithms for solving the tridiagonal systems resulting from the algorithm involved like the TDMA method, multigrid method, classical or preconditioned conjugate gradient method, and finally the parallelization of the algorithms previously mentioned and in particular the subdomain method with synchronous or asynchronous communication between the parallel processes. Finally, we consider also non-linear situations where, on the one hand, the linear part is perturbed by an increasing diagonal operator and, on the other hand, the solution of the damped wave equation is subject to inequality constraints, leading to the solution of a multivalued system. Note that in section 4, an estimated energy is also established, which in fact establishes that the solution of the problems under consideration depends continuously of the problem data and that these problems are therefore well-posed in the Hadamard sense. Moreover, classically these energy estimates make it possible to show uniqueness of the solution (see [5]). Section 5 presents simulation results limited only to the case of one-dimensional problems. Section 6 concludes this study and suggests possible extensions. Finally, in section 7, a technical appendix on the calculation of the eigenelements of the discretization matrices of the convection-diffusion problem in square or cubic domains is detailed. Thus, these results are used to establish the stability condition of the proposed numerical schemes in section 4.

2 Damped wave equation equipped with homogeneous Dirichlet boundary conditions.

In what follows let $L = 1.0$ and T be two positive real numbers. Consider the following one-dimensional damped wave equation:

$$\begin{cases} \mu \cdot \frac{\partial^2 p(x,t)}{\partial t^2} - \theta \cdot \frac{\partial^2 p(x,t)}{\partial x^2} + \mu_1 \cdot \frac{\partial p(x,t)}{\partial t} + \delta \cdot p(x,t) = \bar{g}(x,t), \text{ on } [0, L] \times [0, T], \\ p(0,t) = p(L,t) = 0, \text{ for } t \geq 0, \\ p(x,0) = p_0(x) \text{ and } \frac{\partial p(x,0)}{\partial t} = p_1(x) \text{ for } x \in [0, L], \end{cases} \quad (3)$$

where μ, θ, μ_1 are positive constants and $\delta \geq 0$. In the sequel, we consider the following change of variables

$$c^2 = \frac{\theta}{\mu}, d = \frac{\mu_1}{\mu}, a = \frac{\delta}{\mu}, g(x,t) = \frac{\bar{g}(x,t)}{\mu}, \quad (4)$$

such that problem (3) can now be written as follows:

$$\begin{cases} \frac{\partial^2 p(x,t)}{\partial t^2} - c^2 \cdot \frac{\partial^2 p(x,t)}{\partial x^2} + d \cdot \frac{\partial p(x,t)}{\partial t} + a \cdot p(x,t) = g(x,t), \text{ on } [0, L] \times [0, T], \\ p(0,t) = p(L,t) = 0, \text{ for } t \geq 0, \\ p(x,0) = p_0(x) \text{ and } \frac{\partial p(x,0)}{\partial t} = p_1(x) \text{ for } x \in [0, L]. \end{cases} \quad (5)$$

2.1 Energy inequality.

For the homogeneous problem and when $a = 0$, let us cite a first result of total energy decrease.

Proposition 1 *When $a = 0$, let us consider the homogeneous boundary value problem associated to (5). Assume that p is a sufficiently regular solution on $[0, L] \times [0, T]$ of the homogeneous boundary value problem associated to (5). Then, the following energy equality*

$$E = \left(\int_0^L \left| \frac{\partial p}{\partial t} \right|^2 dx + c^2 \int_0^L \left| \frac{\partial p}{\partial x} \right|^2 dx \right) \quad (6)$$

is decreasing.

Proof 1 *Indeed let us multiply the homogeneous problem by $\frac{\partial p}{\partial t}$ and integrate with respect to x on the interval $[0, L]$; so we obtain*

$$\int_0^L \frac{\partial^2 p}{\partial t^2} \cdot \frac{\partial p}{\partial t} dx + \int_0^L \left(\frac{\partial p}{\partial t} \right)^2 dx - c^2 \int_0^L \frac{\partial^2 p}{\partial x^2} \cdot \frac{\partial p}{\partial t} dx = 0,$$

by exchanging the operators for integration in space and derivation in time, which is permissible for regular p , and since $\int_0^L \left(\frac{\partial p}{\partial t} \right)^2 dx \geq 0$, we get:

$$\frac{1}{2} \cdot \frac{d}{dt} \left(\int_0^L \left| \frac{\partial p}{\partial t} \right|^2 dx \right) - c^2 \int_0^L \frac{\partial^2 p}{\partial x^2} \cdot \frac{\partial p}{\partial t} dx \leq 0;$$

we can also write:

$$\int_0^L \frac{\partial^2 p}{\partial x^2} \cdot \frac{\partial p}{\partial t} dx = \left[\frac{\partial p}{\partial x} \cdot \frac{\partial p}{\partial t} \right]_0^L - \int_0^L \frac{\partial p}{\partial x} \cdot \frac{d}{dx} \left(\frac{\partial p}{\partial t} \right) dx = - \int_0^L \frac{\partial p}{\partial x} \cdot \frac{d}{dx} \left(\frac{\partial p}{\partial t} \right) dx;$$

indeed $\left[\frac{\partial p}{\partial x} \cdot \frac{\partial p}{\partial t} \right]_0^L = 0$ since $p(0, t) = p(L, t) = 0$ leads to $\frac{\partial p(0, t)}{\partial t} = \frac{\partial p(L, t)}{\partial t} = 0$. Consequently, we obtain finally:

$$\frac{1}{2} \cdot \frac{d}{dt} \left(\int_0^L \left| \frac{\partial p}{\partial t} \right|^2 dx \right) + \frac{c^2}{2} \frac{d}{dt} \left(\int_0^L \left| \frac{\partial p}{\partial x} \right|^2 dx \right) \leq 0, \quad (7)$$

and the derivative with respect to the time of the energy E is nonpositive, so the proof is complete.

Remark 1 *Note that:*

$$E_c = \left(\int_0^L \left| \frac{\partial p}{\partial t} \right|^2 dx \right)$$

represents the kinetic energy of the physical system while

$$E_{int} = c^2 \left(\int_0^L \left| \frac{\partial p}{\partial x} \right|^2 dx \right)$$

represents the intern energy. Then, $E = E_c + E_{int}$ and with no external force applied, the total energy E is decreasing.

Remark 2 Proposition 1 is stated when $a \equiv 0$. When $a \neq 0$ by a similar way, i.e. by multiplying the homogeneous problem by $\frac{\partial p}{\partial t}$ and by integrating with respect to x on the interval $[0, L]$, we have to take into account of the term:

$$a \cdot \int_0^L p(x, t) \cdot \frac{\partial p(x, t)}{\partial t} dx = \frac{a}{2} \int_0^L \frac{\partial p^2(x, t)}{\partial t} dx = \frac{a}{2} \cdot \frac{d}{dt} \left(\int_0^L p^2(x, t) dx \right).$$

Thus, in the situation where $a \neq 0$, due to the fact that once again $\int_0^L \left(\frac{\partial p}{\partial t}\right)^2 dx \geq 0$ and according to inequality (7) the result of Proposition 1 can be extended since the term

$$\mathcal{E} = \left(\int_0^L \left| \frac{\partial p}{\partial t} \right|^2 dx + c^2 \int_0^L \left| \frac{\partial p}{\partial x} \right|^2 dx \right) + a \int_0^L p^2(x, t) dx$$

is obviously decreasing; indeed the derivative of \mathcal{E} with respect to the time is nonpositive.

When $a = 0$, let us consider now the situation where an external force is applied to the system. In this case we have the following result:

Proposition 2 When $a = 0$, let us consider the boundary value problem (5). Assume that p is a sufficiently regular solution on $[0, L] \times [0, T]$ of problem (5). Then, the following inequality of energy holds:

$$E_{total}(t) \leq (E_{total}(0) + E_{externalforce}) \cdot e^t, \quad (8)$$

where $E_{total}(t)$ is the total energy at time t defined by:

$$E_{total}(t) = \int_0^L \left(\left| \frac{\partial p(x, t)}{\partial t} \right|^2 + c^2 \left| \frac{\partial p(x, t)}{\partial x} \right|^2 \right) dx,$$

$E_{total}(0)$ represents the total energy at the initial time $t = 0$ and $E_{externalforce}$ is the energy due to the external forces defined by

$$E_{externalforce} = \int_0^t \int_0^L g(x, t)^2 dx \cdot dt.$$

Proposition 2 can be established by using the classic Gronwall Lemma [6].

Lemma 1 Let $t \rightarrow \phi(t)$ be a nonnegative regular continuous function satisfying

$$\begin{cases} \phi(t) \leq A + B \int_0^t \phi(\tau) d\tau \\ \phi(0) \leq A \end{cases} \quad (9)$$

where A and B are two positive real numbers. Then, for all $t \geq 0$ we have:

$$\phi(t) \leq A \cdot e^{Bt}.$$

Proof 2 In a similar way to the proof of the Proposition 1, let us multiply equation (5) by $\frac{\partial p}{\partial t}$ and integrate with respect to x on the interval $[0, L]$. Concerning the left member obtained in

the homogeneous case, the previous estimate are unchanged. Thus, we obtain the following inequality:

$$\frac{1}{2} \cdot \frac{d}{dt} \left(\int_0^L \left| \frac{\partial p}{\partial t} \right|^2 dx + c^2 \int_0^L \left| \frac{\partial p}{\partial x} \right|^2 dx \right) \leq \int_0^L g \cdot \frac{\partial p}{\partial t} dx \leq \frac{1}{2} \left(\int_0^L g^2 dx + \int_0^L \left| \frac{\partial p}{\partial t} \right|^2 dx \right),$$

since $\gamma \cdot \xi \leq \frac{1}{2} \cdot (\gamma^2 + \xi^2)$. Then,

$$\frac{d}{dt} \left(\int_0^L \left| \frac{\partial p}{\partial t} \right|^2 dx + c^2 \int_0^L \left| \frac{\partial p}{\partial x} \right|^2 dx \right) \leq \int_0^L g^2 dx + \int_0^L \left| \frac{\partial p}{\partial t} \right|^2 dx.$$

Let us now integrate the previous relation with respect to the time on the interval $[0, t]$; then we obtain:

$$\begin{aligned} \int_0^L \left| \frac{\partial p(x, t)}{\partial t} \right|^2 dx + c^2 \int_0^L \left| \frac{\partial p(x, t)}{\partial x} \right|^2 dx &\leq \int_0^L \left| \frac{\partial p(x, 0)}{\partial t} \right|^2 dx + c^2 \int_0^L \left| \frac{\partial p(x, 0)}{\partial x} \right|^2 dx \\ &+ \int_0^t \int_0^L g^2(x, t) dx \cdot dt + \int_0^t \int_0^L \left| \frac{\partial p(x, t)}{\partial t} \right|^2 dx \cdot dt. \end{aligned} \quad (10)$$

The previous inequality does not change by adding the following positive term to the second member $c^2 \int_0^t \int_0^L \left| \frac{\partial p}{\partial x} \right|^2 dx \cdot dt$. Hence, let

$$\phi(t) = \int_0^L \left(\left| \frac{\partial p(x, t)}{\partial t} \right|^2 + c^2 \left| \frac{\partial p(x, t)}{\partial x} \right|^2 \right) dx,$$

such that inequality (10) can be written as follows:

$$\phi(t) \leq \phi(0) + \int_0^t \int_0^L g^2(x, t) dx \cdot dt + \int_0^t \phi(\tau) d\tau \equiv A + \int_0^t \phi(\tau) d\tau,$$

where $A = \phi(0) + \int_0^t \int_0^L g^2(x, t) dx \cdot dt$. Then, by applying the Gronwall lemma recalled above, we obtain:

$$\phi(t) \leq A \cdot e^t = \left(\phi(0) + \int_0^t \int_0^L g^2 dx \cdot dt \right) \cdot e^t$$

and the proof is complete.

Remark 3 The previous result shows a continuous dependence on initial conditions and energy input. However, the result obtained in Proposition 2 is not as sharp as the one obtained in the homogeneous case and presented in Proposition 1.

Remark 4 Once again, let us consider the case where $a \neq 0$. In this case, according to the result stated in Remark 2, the following inequality holds:

$$\frac{1}{2} \cdot \frac{d}{dt} \left(\int_0^L \left| \frac{\partial p}{\partial t} \right|^2 dx + c^2 \int_0^L \left| \frac{\partial p}{\partial x} \right|^2 dx + a \int_0^L p(x, t)^2 dx \right) \leq \frac{1}{2} \left(\int_0^L g^2 dx + \int_0^L \left| \frac{\partial p}{\partial t} \right|^2 dx \right).$$

Thus, in a similar way concerning the processing of the right hand side of the previous inequality, if we denote by:

$$\psi(t) = \int_0^L (|\frac{\partial p}{\partial t}|^2 + c^2 |\frac{\partial p}{\partial x}|^2 + ap^2) dx,$$

thus, by using once again of the Gronwall lemma, we obtain the following estimation:

$$\psi(t) \leq B.e^t = (\psi(0) + \int_0^t \int_0^L g^2 dx . dt) . e^t$$

Remark 5 The previous inequalities show that the $L^2([0, L] \times [0, T])$ -norm of the solution is bounded as long as we consider the damped wave equation over a finite horizon $[0, T]$.

Remark 6 Thus, we have obtained a continuous dependence of the data occurring in the problem. Consequently, by admitting the existence of the solution, this property shows by means of classical absurd reasoning the unicity of the solution.

2.2 First numerical explicit scheme.

Let m be a positive integer and $h = \frac{L}{m+1}$; for the numerical solution of the damped wave equation h is the spatial discretization step. Similarly let N be a positive integer and $k = \frac{T}{N}$ where here k represents the time discretization step used for the numerical solution of the damped wave equation. The following difference equations are classically used to approximate the different derivatives. Indeed, for the second derivative with respect to the time, the following centered finite difference scheme is classically used

$$\frac{\partial^2 p(ih, nk)}{\partial t^2} \approx \frac{p_i^{n+1} - 2p_i^n + p_i^{n-1}}{k^2},$$

while for the first derivative with respect to the time the following classical centered finite difference scheme is also classically used:

$$\frac{\partial p(ih, nk)}{\partial t} \approx \frac{p_i^{n+1} - p_i^{n-1}}{2k};$$

concerning the approximation of the second derivative with respect to the space, the following finite difference scheme is also usually used:

$$\frac{\partial^2 p(ih, nk)}{\partial x^2} \approx \frac{p_{i+1}^n - 2p_i^n + p_{i-1}^n}{h^2}.$$

Thus, using these three previous approximation for $1 \leq i \leq m, n > 1$, we can define an explicite scheme for the numerical solution of the damped wave equation (5):

$$\begin{cases} \frac{p_i^{n+1} - 2p_i^n + p_i^{n-1}}{k^2} + d(\frac{p_i^{n+1} - p_i^{n-1}}{2k}) - c^2(\frac{p_{i+1}^n - 2p_i^n + p_{i-1}^n}{h^2}) + a.p_i^n = g(ih, nk), \\ p_0^n = p_{m+1}^n = 0 \text{ and } p_i^0 = p_0(ih), \frac{p_i^1 - p_i^0}{k} = p_1(ih). \end{cases}$$

Let us multiply the previous relation by k^2 and consider also the following change of variable:

$$\gamma = \frac{d.k}{2}, \alpha = \frac{c.k}{h}, \beta = a.k^2, g_i^n = g(ih, nk), \quad (11)$$

such that the numerical scheme can be written as follows:

$$(\gamma + 1).p_i^{n+1} = k^2.g_i^n + \alpha^2(p_{i+1}^n + p_{i-1}^n) + (2 - 2.\alpha^2 - \beta).p_i^n + (\gamma - 1).p_i^{n-1},$$

or finally

$$\begin{cases} p_i^{n+1} = \frac{k^2}{\gamma+1}.g_i^n + \frac{\alpha^2}{\gamma+1}.(p_{i+1}^n + p_{i-1}^n) + \frac{2-2.\alpha^2-\beta}{\gamma+1}.p_i^n + \frac{\gamma-1}{\gamma+1}.p_i^{n-1}, \\ p_0^n = p_{m+1}^n = 0 \text{ and } p_i^0 = p_0(ih), p_i^1 = p_i^0 + k.p_1(ih). \end{cases} \quad (12)$$

In order to analyze this first numerical scheme, let us calculate the truncation error and determine its numerical stability conditions.

For the previous scheme, we replace p_i^n by $p(x_i, t_n)$ in the previous relation at the point (x_i, t_n) and we define the truncation error F_i^n as the difference between the first and second members of these previous quantities as follows:

$$\begin{aligned} F_i^n &= \frac{p(x_i, t_{n+1}) - 2p(x_i, t_n) + p(x_i, t_{n-1}))}{k^2} - g(x_i, t_n) - d\left(\frac{p(x_i, t_{n+1}) - p(x_i, t_{n-1}))}{2k}\right) \\ &\quad - c^2\left(\frac{p(x_{i+1}, t_n) - 2p(x_i, t_n) + p(x_{i-1}, t_n))}{h^2}\right) - ap(x_i, t_n). \end{aligned}$$

Let us consider the limited expansions of the quantities involved in F_i^n . Thus, we have:

$$\frac{p(x_{i+1}, t_n) - 2p(x_i, t_n) + p(x_{i-1}, t_n))}{h^2} = \frac{\partial^2 p(x_i, t_n)}{\partial x^2} + \frac{h^2}{24}(p_x^{(4)}(\tilde{x}_i, t_n) + p_x^{(4)}(\bar{x}_i, t_n)),$$

where $p_x^{(4)}$ denotes the fourth derivative of p with respect to x and $x_{i-1} < \tilde{x}_i < x_i$ and $x_i < \bar{x}_i < x_{i+1}$. Moreover, with similar notations for the counterpart to time derivative approximation, we obtain:

$$\begin{aligned} \frac{p(x_i, t_{n+1}) - 2p(x_i, t_n) + p(x_i, t_{n-1}))}{k^2} &= \frac{\partial^2 p(x_i, t_n)}{\partial t^2} + \frac{k^2}{24}(p_t^{(4)}(x_i, \tilde{t}_n) + p_t^{(4)}(x_i, \bar{t}_n)), \\ \frac{p(x_i, t_{n+1}) - p(x_i, t_{n-1}))}{2k} &= \frac{\partial p(x_i, t_n)}{\partial t} + \frac{k^2}{3}p_t^{(3)}(x_i, t_n). \end{aligned}$$

Consequently, after simplification and since $p(x_i, t_n)$ is solution of the considered damped wave equation, we obtain:

$$F_i^n = \frac{k^2}{24}(p_t^{(4)}(x_i, \tilde{t}_n) + p_t^{(4)}(x_i, \bar{t}_n)) - \frac{dk^2}{3}.p_t^{(3)}(x_i, t_n) - \frac{c^2 h^2}{24}(p_x^{(4)}(\tilde{x}_i, t_n) + p_x^{(4)}(\bar{x}_i, t_n)).$$

Then taking the absolute value of each partial derivative on $[0, L] \times [0, T]$, we obtain finally the following estimate:

$$|F_i^n| \leq \frac{k^2}{12} Q_t^{(4)} + \frac{dk^2}{3} Q_t^{(3)} + \frac{c^2 h^2}{12} Q_x^{(4)},$$

where $Q_t^{(4)}$ denotes the maximum of the absolute value of the fourth derivative of p with respect to t , $Q_t^{(3)}$ denotes also the maximum absolute value of the third derivative of p with respect to t and accordingly for $Q_x^{(4)}$. Finally, if F denotes the maximum value of $|F_i^n|$ for all i and n , then after appropriate majoration for $n > 1$, we obtain finally:

$$F \leq Q(k^2 + h^2),$$

and we can conclude by stating the following result.

Proposition 3 *The numerical explicit scheme (12) is consistent.*

Remark 7 *For $n > 1$ the numerical scheme (12) is of order 2 in space and time. For $n = 1$ this scheme is of order 1 in time and of order 2 in space.*

In order to analyze the numerical stability of the numerical scheme (12), let us use the Von Neumann method. Using the considered notations, for $n > 1$, we can classically write only the homogeneous semidiscretized scheme in time as follows:

$$\begin{aligned} \frac{p^{n+1}(x) - 2p^n(x) + p^{n-1}(x)}{k^2} - c^2 \frac{p^n(x+h) - 2p^n(x) + p^n(x-h)}{h^2} \\ + d \frac{p^{n+1}(x) - p^{n-1}(x)}{2k} + a.p^n(x) = 0. \end{aligned}$$

Taking the Fourier transform of the previous numerical relation, defined by:

$$\hat{p}(l) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-jlx} p(x) dx, \quad j^2 = -1,$$

and then satisfying:

$$e^{\pm jlh} \hat{p}(l) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-jlx} p(x \pm h) dx,$$

we finally obtain:

$$\frac{\hat{p}^{n+1}(l) - 2\hat{p}^n(l) + \hat{p}^{n-1}(l)}{k^2} + d \frac{\hat{p}^{n+1}(l) - \hat{p}^{n-1}(l)}{2k} - c^2 \frac{e^{jlh} - 2 + e^{-jlh}}{h^2} \hat{p}^n(l) + a.\hat{p}^n(l) = 0.$$

Let us multiply by k^2 and using the change of variable (11), so we obtain:

$$\hat{p}^{n+1}(l) - 2\hat{p}^n(l) + \hat{p}^{n-1}(l) + \gamma(\hat{p}^{n+1}(l) - \hat{p}^{n-1}(l)) - \alpha^2(e^{jlh} - 2 + e^{-jlh})\hat{p}^n(l) + \beta\hat{p}^n(l) = 0.$$

Since $e^{jlh} + e^{-jlh} = 2 \cos(lh)$ and $2(\cos(lh) - 1) = -4 \sin^2(\frac{lh}{2})$, we obtain:

$$\hat{p}^{n+1}(l) = -\frac{2(2\alpha^2 \sin^2(\frac{lh}{2}) - 1 + \frac{\beta}{2})}{1 + \gamma} \hat{p}^n(l) - \frac{1 - \gamma}{1 + \gamma} \hat{p}^{n-1}(l),$$

and since obviously $\hat{p}^n(l) \equiv \hat{p}^n(l)$, finally it follows:

$$\begin{pmatrix} \hat{p}^{n+1}(l) \\ \hat{p}^n(l) \end{pmatrix} = \begin{pmatrix} -\frac{2(2\alpha^2 \sin^2(\frac{lh}{2}) - 1 + \frac{\beta}{2})}{1 + \gamma} & -\frac{1 - \gamma}{1 + \gamma} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \hat{p}^n(l) \\ \hat{p}^{n-1}(l) \end{pmatrix}.$$

Let

$$\mathcal{B} = \begin{pmatrix} -\frac{2(2\alpha^2 \sin^2(\frac{lh}{2}) - 1 + \frac{\beta}{2})}{1 + \gamma} & -\frac{1 - \gamma}{1 + \gamma} \\ 1 & 0 \end{pmatrix}$$

be the amplification matrix. We know classically that a stability condition is given when the eigenvalues of \mathcal{B} are of modulus less than one (see [6]). The characteristic equation of \mathcal{B} is given by:

$$\lambda^2 + \frac{2(2\alpha^2 \sin^2(\frac{lh}{2}) - 1 + \frac{\beta}{2})}{1 + \gamma} \lambda + \frac{1 - \gamma}{1 + \gamma} = 0.$$

The product of the roots is equal to $\frac{1 - \gamma}{1 + \gamma}$ and the modulus of the two eigenvalues of \mathcal{B} are less than one when the sum of the two eigenvalues is less than two. This leads to:

$$2\alpha^2 \sin^2(\frac{lh}{2}) - 1 + \frac{\beta}{2} \leq 1 + \gamma.$$

Then, the numerical scheme is stable in the Von Neumann sense if each following inequality are satisfied, i.e. if:

$$2\alpha^2 \sin^2(\frac{lh}{2}) - 1 \leq 1 \text{ and } \frac{\beta}{2} \leq \gamma,$$

which leads to

$$\alpha \leq 1 \text{ and } \frac{\beta}{2} \leq \gamma,$$

since obviously $\sin^2(\frac{lh}{2}) \leq 1$. Returning to the initial notations of problem (5), it follows:

$$\frac{ck}{h} \leq 1 \text{ and } ak \leq d.$$

The first relation corresponds to the Courant-Friedrichs-Levy condition (C.F.L.) which must be classically also verified for the stability of numerical schemes applied for the solution of the wave equation.

Note also that the stability condition can be obtained by using a different method. Indeed, we consider the one-dimensional Laplace equation with homogeneous Dirichlet boundary conditions:

$$\begin{cases} \frac{d^2 q(x)}{dx^2} = f(x), \text{ on } [0, L], \\ q(0) = q(L) = 0. \end{cases}$$

The discretization of the previous problem by the finite difference method leads classically to the resolution of a linear system where the discretization matrix \mathcal{A} is symmetric and tridiagonal; the entries $a_{i,j}$ of \mathcal{A} are given by:

$$a_{i,j} = \begin{cases} a_{i,i} = 2.0 \text{ for } 1 \leq i \leq m, \\ a_{i,i\pm 1} = -1.0, \\ a_{i,j} = 0 \text{ for all } (i,j) \text{ such that } |i-j| > 1. \end{cases}$$

The matrix \mathcal{A} being symmetric and consequently diagonalizable, the eigenvalues of \mathcal{A} are given by (see annex 7.1):

$$\lambda_l(\mathcal{A}) = 4 \sin^2\left(\frac{l\pi h}{2}\right), l = 1, \dots, m,$$

while the components of the associated eigenvectors are given by:

$$v_i^{(l)}(\mathcal{A}) = \sin(il\pi h), i, l = 1, \dots, m,$$

where, in the present situation, m is the size of \mathcal{A} . Let us denote by Λ the diagonal matrix constituted by the eigenvalues of \mathcal{A} ; so classically, we have:

$$\Lambda = S^{-1} \mathcal{A} S,$$

with S is the matrix where each column is constituted by each eigenvector.

Consider the homogeneous scheme (12) which can be written vectorially as follows:

$$P^{n+1} = \frac{1.0}{\gamma + 1} (2\mathcal{I} - \alpha^2 \mathcal{A} - \beta \mathcal{I}) P^n + \frac{\gamma - 1}{\gamma + 1} P^{n-1}.$$

Let $\bar{P}^n = S P^n$; then the previous relation can be written as follows:

$$\bar{P}^{n+1} = \frac{1.0}{\gamma + 1} (2\mathcal{I} - \alpha^2 \Lambda - \beta \mathcal{I}) \bar{P}^n + \frac{\gamma - 1}{\gamma + 1} \bar{P}^{n-1}.$$

As previously, in the Von Neumann analysis, we can write:

$$\begin{pmatrix} \bar{P}^{n+1} \\ \bar{P}^n \end{pmatrix} = \begin{pmatrix} \frac{1.0}{\gamma+1} (2\mathcal{I} - \alpha^2 \Lambda - \beta \mathcal{I}) & \frac{\gamma-1}{1+\gamma} I \\ I & 0 \end{pmatrix} \begin{pmatrix} \bar{P}^n \\ \bar{P}^{n-1} \end{pmatrix}.$$

V being an eigenvector of the matrix \mathcal{A} , let us search \bar{P}^n of the form $\bar{P}^n = \lambda V$, where λ is classically the eigenvalue of the matrix:

$$\begin{pmatrix} \frac{1.0}{\gamma+1} (2\mathcal{I} - \alpha^2 \Lambda - \beta \mathcal{I}) & \frac{\gamma-1}{1+\gamma} I \\ I & 0 \end{pmatrix};$$

thus, λ satisfy the second-degree equation:

$$\lambda^2 + \frac{(4\alpha^2 \sin^2(\frac{l\pi h}{2}) - 2 + \beta)}{1 + \gamma} \lambda + \frac{1 - \gamma}{1 + \gamma} = 0,$$

and, we are now reduced to an analogous framework similar to the one used for stability analysis studied by the Von Neumann method. Consequently, using the same calculations a necessary stability condition is also given by:

$$\alpha \leq 1 \text{ and } \frac{\beta}{2} \leq \gamma.$$

Since the numerical scheme is consistent and stable, we have a convergence result when the discretization steps in time k and space h tend towards zero (see [5] [7] and [8]).

Proposition 4 *The numerical scheme (12), being consistent and stable when*

$$\alpha \leq 1 \text{ and } \frac{\beta}{2} \leq \gamma,$$

is convergent when $k \rightarrow 0$ and $h \rightarrow 0$.

Therefore, the previous result expresses that the stability and the consistency are a necessary condition for convergence.

Remark 8 *Let us express the stability conditions for the initial problem (3) considering the change of variable (4). We immediately obtain the following conditions:*

$$\sqrt[2]{\frac{\theta}{\mu}} \frac{k}{h} \leq 1 \text{ and } \delta k \leq \mu_1.$$

2.3 Second numerical explicit scheme.

For the numerical solution of problem (5), let us now consider another time-explicit numerical scheme where the first derivative with respect to the time is discretized by a decentered scheme. In this case, we have:

$$\frac{\partial p(ih, nk)}{\partial t} \approx \frac{p_i^{n+1} - p_i^n}{k}.$$

Thus, for $1 \leq i \leq m$ and $n > 1$, we can define an explicite scheme for the numerical solution of the damped wave equation (5) as follows:

$$\begin{cases} \frac{p_i^{n+1} - 2p_i^n + p_i^{n-1}}{k^2} + d(\frac{p_i^{n+1} - p_i^n}{k}) - c^2(\frac{p_{i+1}^n - 2p_i^n + p_{i-1}^n}{h^2}) + ap_i^n = g_i^n, \\ p_0^n = p_{m+1}^n = 0 \text{ and } p_i^0 = p_0(ih), \frac{p_i^1 - p_i^0}{k} = p_1(ih). \end{cases}$$

Let us multiply the previous relation by k^2 and consider the change of variable (11) in which γ is replaced by:

$$\bar{\gamma} = dk, \tag{13}$$

such that the numerical scheme can be written as follows:

$$(\bar{\gamma} + 1)p_i^{n+1} = k^2 g_i^n + \alpha^2(p_{i+1}^n + p_{i-1}^n) + (\bar{\gamma} + 2 - 2\alpha^2 - \beta)p_i^n - p_i^{n-1},$$

and finally for $1 \leq i \leq m$ and $n > 1$, we obtain:

$$\begin{cases} p_i^{n+1} = \frac{k^2}{\bar{\gamma}+1}g_i^n + \frac{\alpha^2}{\bar{\gamma}+1}(p_{i+1}^n + p_{i-1}^n) + \frac{\bar{\gamma}+2-2\alpha^2-\beta}{\bar{\gamma}+1}p_i^n - \frac{1}{\bar{\gamma}+1}p_i^{n-1}, \\ p_0^n = p_{m+1}^n = 0 \text{ and } p_i^0 = p_0(ih), p_i^1 = p_i^0 + k.p_1(ih). \end{cases} \quad (14)$$

As previously performed, let us calculate once again the truncation error and determine the numerical stability conditions of the scheme. In this case, for the approximation of the first derivative with respect to the time, the following relation is classically valid:

$$\frac{\partial p(x_i, t_n)}{\partial t} = \frac{p(x_i, t_{n+1}) - p(x_i, t_n)}{k} + \frac{k}{2}p_t^{(2)}(x_i, t_n).$$

Consequently, after simplification since $p(x_i, t_n)$ is solution of the considered damped wave equation, we obtain:

$$F_i^n = \frac{k^2}{24}(p_t^{(4)}(x_i, \tilde{t}_n) + p_t^{(4)}(x_i, \bar{t}_n)) - \frac{dk}{2}.p_t^{(2)}(x_i, t_n) - \frac{c^2 h^2}{24}(p_x^{(4)}(\tilde{x}_i, t_n) + p_x^{(4)}(\bar{x}_i, t_n)).$$

Then, by taking the absolute value of each partial derivative on $[0, L] \times [0, T]$, we obtain finally the following estimate:

$$|F_i^n| \leq \frac{k^2}{12}Q_t^{(4)} + \frac{dk}{2}Q_t^{(2)} + \frac{c^2 h^2}{12}Q_x^{(4)},$$

where $Q_t^{(2)}$ denotes the maximum absolute value of the second derivative of p with respect to t . Finally, if F denotes the maximum value of $|F_i^n|$, for all i and n , we obtain after appropriate majoration:

$$F \leq Q(k + h^2),$$

and we can conclude by stating the following result:

Proposition 5 *The numerical explicit scheme (14) is consistent.*

Remark 9 *For $n \geq 1$ the numerical scheme (14) is of order 1 in time and of order 2 in space.*

Let us now analyze the numerical stability of the scheme (14) by using the Von Neumann method. For $n > 1$, we can write once again the homogeneous semidiscretized scheme in time as follows:

$$\frac{p^{n+1}(x) - 2p^n(x) + p^{n-1}(x)}{k^2} + d \frac{p^{n+1}(x) - p^n(x)}{k} - c^2 \frac{p^n(x+h) - 2p^n(x) + p^n(x-h)}{h^2} + ap^n(x) = 0,$$

Taking the Fourier transform of the previous scheme leads to:

$$\frac{\hat{p}^{n+1}(l) - 2\hat{p}^n(l) + \hat{p}^{n-1}(l)}{k^2} + d \frac{\hat{p}^{n+1}(l) - \hat{p}^n(l)}{k} - c^2 \frac{e^{jlh} - 2 + e^{-jlh}}{h^2} \hat{p}^n(l) + a\hat{p}^n(l) = 0;$$

by multiplying by k^2 the previous relation and by using the change of variable (11) and (13), we obtain:

$$\hat{p}^{n+1}(l) - 2\hat{p}^n(l) + \hat{p}^{n-1}(l) + \bar{\gamma}(\hat{p}^{n+1}(l) - \hat{p}^n(l)) - \alpha^2(e^{jlh} - 2 + e^{-jlh})\hat{p}^n(l) + \beta\hat{p}^n(l) = 0.$$

Since $e^{jlh} + e^{-jlh} = 2 \cdot \cos(lh)$ and $2 \cdot (\cos(lh) - 1) = -4 \cdot \sin^2(\frac{lh}{2})$, we have:

$$\hat{p}^{n+1}(l) = \frac{2 + \bar{\gamma} - 4\alpha^2 \sin^2(\frac{lh}{2}) - \beta}{1 + \bar{\gamma}} \hat{p}^n(l) - \frac{1}{1 + \bar{\gamma}} \hat{p}^{n-1}(l),$$

and finally as seen above, we obtain:

$$\begin{pmatrix} \hat{p}^{n+1}(l) \\ \hat{p}^n(l) \end{pmatrix} = \begin{pmatrix} \frac{2 + \bar{\gamma} - 4\alpha^2 \sin^2(\frac{lh}{2}) - \beta}{1 + \bar{\gamma}} - \frac{1}{1 + \bar{\gamma}} \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} \hat{p}^n(l) \\ \hat{p}^{n-1}(l) \end{pmatrix},$$

where

$$\mathcal{B} = \begin{pmatrix} \frac{2 + \bar{\gamma} - 4\alpha^2 \sin^2(\frac{lh}{2}) - \beta}{1 + \bar{\gamma}} - \frac{1}{1 + \bar{\gamma}} \\ 1 \\ 0 \end{pmatrix}$$

is the amplification matrix associated to the scheme. We know that a stability condition is given when the eigenvalues of \mathcal{B} are of modulus less than one. The characteristic equation associated to \mathcal{B} is given by:

$$\lambda^2 - \frac{2 + \bar{\gamma} - 4\alpha^2 \sin^2(\frac{lh}{2}) - \beta}{1 + \bar{\gamma}} \lambda + \frac{1}{1 + \bar{\gamma}} = 0. \quad (15)$$

The product of the roots is equal to $\frac{1}{1 + \bar{\gamma}} < 1$ and the modulus of the two eigenvalues of \mathcal{B} are less than one when the sum of the two eigenvalues is less than two which leads to:

$$4\alpha^2 \sin^2(\frac{lh}{2}) + \beta \leq 4 + 3\bar{\gamma},$$

since β and $\bar{\gamma}$ are positive numbers. Then, the numerical scheme is stable in the Von Neumann sense, if each of the following inequality are satisfied:

$$4\alpha^2 \sin^2(\frac{lh}{2}) \leq 4 \text{ and } \beta \leq 3\bar{\gamma},$$

this leads to

$$\alpha \leq 1 \text{ and } \beta \leq 3\bar{\gamma},$$

since obviously $\sin^2(\frac{lh}{2}) \leq 1$. Returning to the initial notations of problem (5) the numerical stability of the numerical scheme (14) is guaranteed when:

$$\frac{ck}{h} \leq 1 \text{ and } \frac{\beta}{3} = \frac{ak^2}{3} \leq \bar{\gamma} = dk$$

which, by using the initial notations defined in problem (3), leads to:

$$\sqrt[2]{\frac{\theta}{\mu}} \frac{k}{h} \leq 1 \text{ and } \frac{\delta k}{3} \leq \mu_1.$$

Remark 10 Note that for the numerical scheme (14), we can consider a time step k greater than the one chosen for the numerical scheme (12), in fact three times larger which is very much appreciated. However, in this case, the precision is less good since the truncation error is in $O(k)$.

Note also that, once again, the stability condition can be obtained by using the matricial method considered in subsection 2.2. Hence, by considering the vectorial formulation of the homogeneous scheme (14),

$$P^{n+1} = \frac{1}{\bar{\gamma} + 1} ((\bar{\gamma} + 2 - \beta)I - \alpha^2 A) P^n - \frac{1}{\bar{\gamma} + 1} P^{n-1},$$

in the eigenvector basis, the above relation can be written as follows:

$$\bar{P}^{n+1} = \frac{1}{\bar{\gamma} + 1} ((\bar{\gamma} + 2 - \beta)I - \alpha^2 \Lambda) \bar{P}^n - \frac{1}{\bar{\gamma} + 1} \bar{P}^{n-1},$$

and once again, we can write:

$$\begin{pmatrix} \bar{P}^{n+1} \\ \bar{P}^n \end{pmatrix} = \begin{pmatrix} \frac{1}{\bar{\gamma} + 1} ((\bar{\gamma} + 2 - \beta)I - \alpha^2 \Lambda) - \frac{1}{\bar{\gamma} + 1} I \\ I \end{pmatrix} \begin{pmatrix} \bar{P}^n \\ \bar{P}^{n-1} \end{pmatrix}.$$

Thus, since the eigenvalues of the matrix \mathcal{A} are given by:

$$\lambda_l(\mathcal{A}) = 4 \sin^2\left(\frac{l\pi h}{2}\right), l = 1, \dots, m,$$

then the eigenvalues of the amplification matrix is solution of the second-degree equation

$$\lambda^2 - \frac{(\bar{\gamma} + 2 - \beta) - 4\alpha^2 \sin^2\left(\frac{l\pi \cdot h}{2}\right)}{\bar{\gamma} + 1} \lambda + \frac{1}{\bar{\gamma} + 1} = 0.$$

This equation is similar to the characteristic equation (15). Consequently, we can deduce similar conditions of numerical stability.

Concerning the convergence we can conclude similar results than the one stated in Proposition 4.

Proposition 6 When the numerical scheme (14) is consistent and stable, i.e.

$$\alpha \leq 1 \text{ and } \frac{\beta}{3} \leq \bar{\gamma},$$

then it is convergent when $k \rightarrow 0$ and $h \rightarrow 0$.

2.4 First numerical implicit scheme.

Similarly to the discretization considered in subsection 2.2 let us consider the approximation of the first derivative with respect to the time by a centered finite difference scheme. Hence, we consider now the following implicit time marching numerical scheme for the solution of the damped wave equation equipped with homogeneous Dirichlet boundary conditions. Thus, for $n > 1$ and $1 \leq i \leq m$, we have:

$$\begin{cases} \frac{p_i^{n+1} - 2p_i^n + p_i^{n-1}}{k^2} + d\left(\frac{p_i^{n+1} - p_i^{n-1}}{2k}\right) - c^2\left(\frac{p_{i+1}^{n+1} - 2p_i^{n+1} + p_{i-1}^{n+1}}{h^2}\right) + ap_i^{n+1} = g(ih, (n+1)k), \\ p_0^n = p_{m+1}^n = 0 \text{ and } p_i^0 = p_0(ih), \frac{p_i^1 - p_i^0}{k} = p_1(ih). \end{cases} \quad (16)$$

Note that the truncation error is the same as in subsection 2.2, i.e.

$$F \leq Q(k^2 + h^2),$$

and we can conclude by stating the following result:

Proposition 7 *The numerical implicit scheme (16) is consistent and for $n > 1$ and is of order 2 in space and in time.*

Let us multiply relation (16) by k^2 and use the change of variable (11). Thus, the studied numerical scheme can be written as follows for $n > 1$ and $1 \leq i \leq m$,

$$(\gamma + 1)p_i^{n+1} - \alpha^2(p_{i+1}^{n+1} + p_{i-1}^{n+1}) + 2\alpha^2 p_i^{n+1} + \beta p_i^{n+1} = k^2 g_i^{n+1} + 2p_i^n + (\gamma - 1)p_i^{n-1},$$

or finally for $n > 1$ and $1 \leq i \leq m$, we have:

$$\begin{cases} -\alpha^2 p_{i-1}^{n+1} + (\gamma + 2\alpha^2 + \beta + 1)p_i^{n+1} - \alpha^2 p_{i+1}^{n+1} = k^2 g_i^{n+1} + 2p_i^n + (\gamma - 1)p_i^{n-1}, \\ p_0^n = p_{m+1}^n = 0 \text{ and } p_i^0 = p_0(ih), p_i^1 = p_i^0 + k.p_1(ih). \end{cases} \quad (17)$$

Thus, at each time step, we have to solve the following linear system:

$$(\alpha^2 A + (\gamma + \beta + 1)I)P^{n+1} = k^2 G^{n+1} + 2P^n + (\gamma - 1)P^{n-1}. \quad (18)$$

Note that, since \mathcal{A} is symmetric positive definite, then the matrix $(\alpha^2 \mathcal{A} + (\gamma + \beta + 1)I)$ is also symmetric positive definite and then invertible. For the solution of such linear system we can use the TDMA algorithm (see [9]), corresponding in fact to an adaptation of the Gauss elimination method for tridiagonal matrices.

Let us now analyse the numerical stability of the time marching scheme (17) by the Von Neumann method. Thus, by considering the associated homogeneous semidiscretized scheme in time, for $n > 1$:

$$\begin{aligned} \frac{p^{n+1}(x) - 2p^n(x) + p^{n-1}(x)}{k^2} + d\left(\frac{p^{n+1}(x) - p^{n-1}(x)}{2k}\right) + ap^{n+1}(x) \\ - c^2\left(\frac{p^{n+1}(x+h) - 2p^{n+1}(x) + p^{n+1}(x-h)}{h^2}\right) = 0, \end{aligned}$$

written as follows:

$$-\alpha^2(p^{n+1}(x+h) + p^{n+1}(x-h)) + (\gamma + 2\alpha^2 + \beta + 1)p^{n+1}(x) = 2p^n(x) + (\gamma - 1)p^{n-1}(x),$$

the Fourier transform of the previous relation leads to:

$$(2\alpha^2(1 - \cos(lh)) + \gamma + \beta + 1)\hat{p}^{n+1}(l) = 2\hat{p}^n(l) + (\gamma - 1)\hat{p}^{n-1}(l),$$

$$(4\alpha^2 \sin^2(\frac{lh}{2}) + \gamma + \beta + 1)\hat{p}^{n+1}(l) = 2\hat{p}^n(l) + (\gamma - 1)\hat{p}^{n-1}(l),$$

then:

$$\hat{p}^{n+1}(l) = \frac{2}{(4\alpha^2 \sin^2(\frac{lh}{2}) + \gamma + \beta + 1)}\hat{p}^n(l) + \frac{(\gamma - 1)}{(4\alpha^2 \sin^2(\frac{lh}{2}) + \gamma + \beta + 1)}\hat{p}^{n-1}(l),$$

which leads to:

$$\begin{pmatrix} \hat{p}^{n+1}(l) \\ \hat{p}^n(l) \end{pmatrix} = \begin{pmatrix} \frac{2}{(4\alpha^2 \sin^2(\frac{lh}{2}) + \gamma + \beta + 1)} & \frac{(\gamma - 1)}{(4\alpha^2 \sin^2(\frac{lh}{2}) + \gamma + \beta + 1)} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \hat{p}^n(l) \\ \hat{p}^{n-1}(l) \end{pmatrix}.$$

Consequently, the eigenvalues of the amplification matrix satisfy the following second degree equation:

$$\lambda^2 - \frac{2}{(4\alpha^2 \sin^2(\frac{lh}{2}) + \gamma + \beta + 1)}\lambda + \frac{(1 - \gamma)}{(4\alpha^2 \sin^2(\frac{lh}{2}) + \gamma + \beta + 1)} = 0.$$

Let us remark that the sum of eigenvalues is equal to $\frac{2}{(4\alpha^2 \sin^2(\frac{lh}{2}) + \gamma + \beta + 1)}$ and is strictly positive. The product of the eigenvalues is equal to $\frac{(1 - \gamma)}{(4\alpha^2 \sin^2(\frac{lh}{2}) + \gamma + \beta + 1)}$. Let us consider two distinct cases :

- if $1 - \gamma > 0 \Leftrightarrow 1 > \gamma$ then the product of eigenvalues is positive and the roots are of the same sign and moreover positive since the sum is also positive. The numerical stability of the scheme is guaranteed if the sum of the modulus of the eigenvalues is less than two, i.e.

$$-2 < \frac{2}{(4\alpha^2 \sin^2(\frac{lh}{2}) + \gamma + \beta + 1)} < 2;$$

we can easily verify that these inequalities are always true since γ and β are positive. Consequently, we can conclude that each eigenvalue has a modulus strictly less than one.

- if $1 - \gamma < 0 \Leftrightarrow 1 < \gamma$ then the roots of the characteristic equation are of opposite sign, but since their sum is positive the larger is positive. Once again, we can verify that the following inequalities:

$$-2 < \frac{2}{(4\alpha^2 \sin^2(\frac{lh}{2}) + \gamma + \beta + 1)} < 2$$

are obviously verified. We can therefore conclude by the following proposition:

Proposition 8 *The numerical implicit scheme (17) is unconditionally stable.*

Let us determine this unconditional stability property by using the matricial method. Indeed, let us consider the homogeneous equation associated to (18). Then P^{n+1} is given by:

$$P^{n+1} = (\alpha^2 A + (\gamma + \beta + 1)I)^{-1}(2P^n + (\gamma - 1)P^{n-1}).$$

The matrix $(A + (\gamma + \beta + 1)I)^{-1}$ is diagonalizable and its eigenvalues are equal to

$$\frac{1}{4\alpha^2 \sin^2(\frac{l\pi h}{2}) + \gamma + \beta + 1}.$$

As previously seen, writing the resulting homogeneous numerical scheme in the eigenvector basis, leads to:

$$\begin{pmatrix} \bar{P}^{n+1} \\ \bar{P}^n \end{pmatrix} = \begin{pmatrix} 2(\Lambda + (\gamma + \beta + 1)I)^{-1} & (\gamma - 1)(\Lambda + (\gamma + \beta + 1)I)^{-1} \\ I & 0 \end{pmatrix} \begin{pmatrix} \bar{P}^n \\ \bar{P}^{n-1} \end{pmatrix}.$$

Since Λ is a diagonal matrix, the amplification matrix can be written as follows:

$$\mathcal{B} = \begin{pmatrix} \frac{2}{4\alpha^2 \sin^2(\frac{l\pi h}{2}) + \gamma + \beta + 1} & \frac{(\gamma - 1)}{4\alpha^2 \sin^2(\frac{l\pi h}{2}) + \gamma + \beta + 1} \\ 1 & 0 \end{pmatrix}$$

Thus, the eigenvalues of \mathcal{B} satisfy the second degree equation:

$$\lambda^2 - \frac{2}{4\alpha^2 \sin^2(\frac{l\pi h}{2}) + \gamma + \beta + 1} \lambda - \frac{(\gamma - 1)}{4\alpha^2 \sin^2(\frac{l\pi h}{2}) + \gamma + \beta + 1} = 0,$$

and by using the same approach as for the Von Neumann analysis, we conclude that the numerical scheme is unconditionally stable.

Concerning the convergence, we can conclude similar results than the ones previously stated.

Proposition 9 *The numerical scheme (17) being consistent and unconditionally stable then it is convergent when $k \rightarrow 0$ and $h \rightarrow 0$.*

2.5 Second numerical implicit scheme.

Consider now an other numerical implicit scheme where in the same way as in the subsection 2.3, the first derivative in time is discretized by a decentered scheme. For $n > 1$ and $1 \leq i \leq m$, we then obtain the following scheme:

$$\begin{cases} \frac{p_i^{n+1} - 2p_i^n + p_i^{n-1}}{k^2} + d(\frac{p_i^{n+1} - p_i^n}{k}) - c^2(\frac{p_{i+1}^{n+1} - 2p_i^{n+1} + p_{i-1}^{n+1}}{h^2}) + ap_i^{n+1} = g_i^{n+1}, \\ p_0^{n+1} = p_{m+1}^{n+1} = 0 \text{ and } p_i^0 = p_0(ih), \frac{p_i^1 - p_i^0}{k} = p_1(ih). \end{cases}$$

Let us multiply the previous relation by k^2 and let again $\bar{\gamma} = dk$ such that the previous numerical scheme can be written as follows for $1 \leq i \leq m$ and $n > 1$:

$$\begin{cases} -\alpha^2 p_{i+1}^{n+1} + (\bar{\gamma} + 1 + 2\alpha^2 + \beta)p_i^{n+1} - \alpha^2 p_{i-1}^{n+1} = k^2 g_i^{n+1} + (2 + \bar{\gamma})p_i^n - p_i^{n-1}, \\ p_0^{n+1} = p_{m+1}^{n+1} = 0 \text{ and } p_i^0 = p_0(ih), p_i^1 = p_i^0 + k.p_1(ih). \end{cases} \quad (19)$$

Then, once again, at each time step, we have to solve the following linear system:

$$((\bar{\gamma} + 1 + \beta)I + \alpha^2 \mathcal{A})P^{n+1} = k^2 G^{n+1} + (2 + \bar{\gamma})P^n - P^{n-1}.$$

Note that the matrix $((\bar{\gamma} + 1 + \beta)I + \alpha^2 \mathcal{A})$ is symmetric positive definite and consequently invertible. For the solution of such linear system, we can use once again the TDMA algorithm. Note that the truncation error is the same than in subsection 2.3, i.e.

$$F \leq Q(k + h^2),$$

and we can conclude by stating the following result:

Proposition 10 *The numerical explicit scheme (19) is consistent and the time marching scheme is of order 1 in time and of order 2 in space.*

For the study of the numerical stability of the numerical implicit scheme (19), consider first the use of the Von Neumann method. Similarly to the previous subsections, we have by considering the homogeneous numerical scheme:

$$-\alpha^2(p^{n+1}(x+h) + p^{n+1}(x-h)) + (\bar{\gamma} + 2\alpha^2 + \beta + 1)p^{n+1}(x) = (2 + \bar{\gamma})p^n(x) - p^{n-1}(x);$$

the Fourier transform of the previous relation leads to:

$$(4\alpha^2 \sin^2(\frac{lh}{2}) + \bar{\gamma} + \beta + 1)\hat{p}^{n+1}(l) = (2 + \bar{\gamma})\hat{p}^n(l) - \hat{p}^{n-1}(l),$$

and then:

$$\hat{p}^{n+1}(l) = \frac{2 + \bar{\gamma}}{4\alpha^2 \sin^2(\frac{lh}{2}) + \bar{\gamma} + \beta + 1} \hat{p}^n(l) - \frac{1}{4\alpha^2 \sin^2(\frac{lh}{2}) + \bar{\gamma} + \beta + 1} \hat{p}^{n-1}(l).$$

As previously, we can deduce the amplification matrix:

$$\mathcal{B} = \begin{pmatrix} \frac{2 + \bar{\gamma}}{4\alpha^2 \sin^2(\frac{lh}{2}) + \bar{\gamma} + \beta + 1} & -\frac{1}{4\alpha^2 \sin^2(\frac{lh}{2}) + \bar{\gamma} + \beta + 1} \\ 1 & 0 \end{pmatrix},$$

and the eigenvalues of \mathcal{B} satisfy the characteristic equation:

$$\lambda^2 - \frac{2 + \bar{\gamma}}{4\alpha^2 \sin^2(\frac{lh}{2}) + \bar{\gamma} + \beta + 1} \lambda + \frac{1}{4\alpha^2 \sin^2(\frac{lh}{2}) + \bar{\gamma} + \beta + 1} = 0.$$

The product of the roots is equal to $\frac{1}{4\alpha^2 \sin^2(\frac{lh}{2}) + \bar{\gamma} + \beta + 1}$ and is positive, so the roots have the same sign and furthermore, this product is less than one. In addition, the sum of the roots is

equal to $\frac{2+\bar{\gamma}}{4\alpha^2 \sin^2(\frac{lh}{2})+\bar{\gamma}+\beta+1}$ and is also positive, so that the roots are positive. The numerical stability of the implicit numerical scheme (19). Hence, we can be stated by a similar way than the one considered in subsection 2.4. Hence, we can easily verify that the following inequalities:

$$-2 \leq \frac{2 + \bar{\gamma}}{4\alpha^2 \sin^2(\frac{lh}{2}) + \bar{\gamma} + \beta + 1} \leq 2$$

are true, which obviously prove that the eigenvalues of \mathcal{B} are of modulus less than one. Therefore, we can conclude that the numerical scheme (19) is unconditionally stable.

Proposition 11 *The numerical implicit scheme (19) is unconditionally stable.*

Let us now verify this property of unconditional stability by the matricial method. Expressed in the eigenvector basis the homogeneous equation associated to the numerical scheme (19) is written as follows:

$$\bar{P}^{n+1} = (\Lambda + (\bar{\gamma} + 1 + \beta)I)^{-1}((2 + \bar{\gamma})\bar{P}^n - \bar{P}^{n-1}),$$

so that the amplification matrix is written as follows:

$$\mathcal{B} = \begin{pmatrix} \frac{2+\bar{\gamma}}{4\alpha^2 \sin^2(\frac{l\pi h}{2})+\bar{\gamma}+\beta+1} & -\frac{1}{4\alpha^2 \sin^2(\frac{l\pi h}{2})+\bar{\gamma}+\beta+1} \\ 1 & 0 \end{pmatrix},$$

and the eigenvalues of \mathcal{B} satisfy the characteristic equation:

$$\lambda^2 - \frac{2 + \bar{\gamma}}{4\alpha^2 \sin^2(\frac{l\pi h}{2}) + \bar{\gamma} + \beta + 1} \lambda + \frac{1}{4\alpha^2 \sin^2(\frac{l\pi h}{2}) + \bar{\gamma} + \beta + 1} = 0,$$

and, as previously seen, the eigenvalues of \mathcal{B} are of modulus less than one, which allows us to recover the property of unconditional stability.

In addition, the implicit numerical scheme (19) being consistent and unconditionally stable, it is convergent. Thus, we can still state the following convergence results

Proposition 12 *The numerical scheme (19) being consistent and unconditionally stable then it is convergent when $k \rightarrow 0$ and $h \rightarrow 0$.*

2.6 Numerical implicit scheme derived from the Gear scheme.

To approximate the first derivative with respect to the time, we now use the following Gear scheme:

$$\frac{3p_i^{n+1} - 4p_i^n + p_i^{n-1}}{2k} = \frac{\partial p(i.h, (n+1).k)}{\partial t} - \frac{k^2}{3} \frac{\partial^3 p(i.h, (n+1).k)}{\partial t^3} + O(k^2).$$

Combining this approximation of the first derivative in time at the point (ih, nk) with the approximations of the second derivatives with respect to the time and to the space, we obtain the following implicit scheme for the damped wave equation for $n > 1$ and $1 \leq i \leq m$:

$$\begin{cases} \frac{p_i^{n+1} - 2p_i^n + p_i^{n-1}}{k^2} + d \left(\frac{3p_i^{n+1} - 4p_i^n + p_i^{n-1}}{2k} \right) - c^2 \left(\frac{p_{i+1}^{n+1} - 2p_i^{n+1} + p_{i-1}^{n+1}}{h^2} \right) + ap_i^{n+1} = g_i^{n+1}, \\ p_0^n = p_{m+1}^n = 0 \text{ and } p_i^0 = p_0(ih), \frac{p_i^1 - p_i^0}{k} = p_1(ih). \end{cases} \quad (20)$$

Since the Gear scheme is of order two with respect to the time and due to the result presented in subsection 2.2, note that the truncation error is given by:

$$F \leq Q(k^2 + h^2),$$

and consequently, for $n > 1$, the new scheme derived from the Gear approximation is of order two in space and time, so that we can conclude by stating the following result.

Proposition 13 *The numerical explicit scheme (20) is consistent.*

Let us multiply by k^2 and by using the change of variable (11), thus the numerical scheme can be written as follows, for $n > 1$ and $1 \leq i \leq m$:

$$\begin{cases} -\alpha^2 p_{i+1}^{n+1} + (3\gamma + \beta + 2\alpha^2 + 1)p_i^{n+1} - \alpha^2 p_{i-1}^{n+1} = k^2 g_i^{n+1} + 2(1 + 2\gamma)p_i^n - (1 + \gamma)p_i^{n-1}, \\ p_0^n = p_{m+1}^n = 0 \text{ and } p_i^0 = p_0(ih), \frac{p_i^1 - p_i^0}{k} = p_1(ih). \end{cases}$$

This above relation can be written vectorially as follows:

$$((3\gamma + \beta + 1)I + \alpha^2 \mathcal{A})P^{n+1} = k^2 G^{n+1} + 2(1 + 2\gamma)P^n - (1 + \gamma)P^{n-1}.$$

So, as previously, we can solve this linear system by using the TDMA method.

Consider now the associated homogeneous semidiscretized scheme in time and let us perform the Fourier transform; this leads to the relation:

$$(4\alpha^2 \sin^2(\frac{lh}{2}) + 3\gamma + \beta + 1)\hat{p}^{n+1} = 2(1 + 2\gamma)\hat{p}^n - (1 + \gamma)\hat{p}^{n-1},$$

or also:

$$\hat{p}^{n+1} = \frac{2(1 + 2\gamma)}{4\alpha^2 \sin^2(\frac{lh}{2}) + 3\gamma + \beta + 1} \hat{p}^n - \frac{1 + \gamma}{4\alpha^2 \sin^2(\frac{lh}{2}) + 3\gamma + \beta + 1} \hat{p}^{n-1}.$$

This relation can also be written in the following form.

$$\begin{pmatrix} \hat{p}^{n+1}(l) \\ \hat{p}^n(l) \end{pmatrix} = \begin{pmatrix} \frac{2(1+2\gamma)}{4\alpha^2 \sin^2(\frac{lh}{2}) + 3\gamma + \beta + 1} & -\frac{1+\gamma}{4\alpha^2 \sin^2(\frac{lh}{2}) + 3\gamma + \beta + 1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \hat{p}^n(l) \\ \hat{p}^{n-1}(l) \end{pmatrix}.$$

Consequently the eigenvalues of the amplification matrix satisfy the following second degree equation:

$$\lambda^2 - \frac{2(1 + 2\gamma)}{4\alpha^2 \sin^2(\frac{lh}{2}) + 3\gamma + \beta + 1} \lambda + \frac{1 + \gamma}{4\alpha^2 \sin^2(\frac{lh}{2}) + 3\gamma + \beta + 1} = 0.$$

The product of the root is positive, so that the eigenvalues have the same sign. In addition the sum of the roots is positive, and thus the eigenvalues are positive. Consequently, the implicit numerical scheme is stable when

$$-2 \leq \frac{2(1 + 2\gamma)}{4\alpha^2 \sin^2(\frac{lh}{2}) + 3\gamma + \beta + 1} \leq 2.$$

We can easily see that these inequalities are always verified and we conclude that the implicit scheme derived from the Gear scheme (20) is unconditionally stable. In addition, since this scheme is consistent, then it is also convergent when $h \rightarrow 0$ and $k \rightarrow 0$.

Once again, we can obtain the same result of unconditional stability by using the matricial method and state the same conclusion of convergence of the numerical implicit scheme when $h \rightarrow 0$ and $k \rightarrow 0$.

2.7 Numerical schemes with variable coefficients of the damped wave equation.

In the previous subsections the coefficients involved in the damped wave equation are assumed to be constant. Consider now the case where these coefficients depend on the space and time variables, i.e. $c \equiv c(x, t)$, $d \equiv d(x, t)$, $a \equiv a(x, t)$, $\gamma \equiv \gamma(x, t)$ (or also $\bar{\gamma}$) and $\beta \equiv \beta(x, t)$. Classically, there are many ways to take into account the fact that these coefficients are not constant. However, for the construction of numerical schemes, we need to ensure that the analysis of the stability of these new schemes is easy. Hence, this is an additional difficulty. Thus, in the sequel, we will simply denote by c_i^n the value of $c(x_i, t^n)$. Similarly, we will use d_i^n , a_i^n , γ_i^n , and β_i^n to denote the values of d , a , γ , β at the point (x_i, t^n) . In this situation, where the coefficients of the damped wave equation are not constant, the matrix method is ineffective for studying the numerical stability of the schemes, since the variable coefficients of the matrices no longer make it possible to simply obtain the expression of the eigenvalues of the discretization matrices. We are therefore required to use the Von Neumann method. Thus, for each time step, we write the above schemes, replacing the constant coefficients by their values at the point (x_i, t^n) and we then consider that, only at this moment t^n , these values of the coefficients are constant. This eliminates the difficulty of performing the Fourier transform with coefficients depending on the space variable.

Therefore, taking into account the approach adopted to study the numerical stability of the previous numerical schemes, it is very simple to establish the stability conditions:

- for the first numerical explicit scheme, the stability is conditional if

$$\max_{i,n} (c_i^n) \frac{k}{h} \leq 1 \text{ and } \max_{i,n} (a_i^n) k \leq \min_{i,n} (d_i^n),$$

- for the second numerical explicit scheme, the stability is conditional if

$$\max_{i,n} (c_i^n) \frac{k}{h} \leq 1 \text{ and } \max_{i,n} (a_i^n) \frac{k}{3} \leq \min_{i,n} (d_i^n),$$

- for the first, the second implicit schemes and also the implicit scheme derived from the Gear scheme, the stability is unconditional.

Remark 11 We refer to [10] for the definition of a three level implicit difference scheme unconditional stable of $O(k^2+h^2)$ for the solution of the damped wave equation with variable coefficients equipped with Dirichlet boundary conditions. In such numerical scheme fictitious points are needed at each time step along the boundary.

3 Damped wave equation equipped with mixed boundary conditions.

Consider the one-dimensional damped wave equation equipped with homogeneous mixed boundary conditions:

$$\begin{cases} \mu \cdot \frac{\partial^2 p(x,t)}{\partial t^2} - \theta \cdot \frac{\partial^2 p(x,t)}{\partial x^2} + \mu_1 \cdot \frac{\partial p(x,t)}{\partial t} + \delta \cdot p(x,t) = \bar{g}(x,t), & \text{on } [0, L] \times [0, T], \\ p(0,t) = 0 \text{ and } \frac{\partial p(L,t)}{\partial x} = 0, & \text{for } t \geq 0, \\ p(x,0) = p_0(x) \text{ and } \frac{\partial p(x,0)}{\partial t} = p_1(x) & \text{for } x \in [0, L], \end{cases} \quad (21)$$

where μ, θ, μ_1 are positive constants and $\delta \geq 0$. In the sequel, we consider once again the change of variables (4) such that problem (21) can now be written as follows:

$$\begin{cases} \frac{\partial^2 p(x,t)}{\partial t^2} - c^2 \cdot \frac{\partial^2 p(x,t)}{\partial x^2} + d \cdot \frac{\partial p(x,t)}{\partial t} + a \cdot p(x,t) = g(x,t), & \text{on } [0, L] \times [0, T], \\ p(0,t) = 0 \text{ and } \frac{\partial p(L,t)}{\partial x} = 0, & \text{for } t \geq 0, \\ p(x,0) = p_0(x) \text{ and } \frac{\partial p(x,0)}{\partial t} = p_1(x) & \text{for } x \in [0, L]. \end{cases} \quad (22)$$

3.1 Energy inequality.

According to the results of subsection 2.1, we have the following result concerning the total energy:

Proposition 14 When $a = 0$ let us consider the homogeneous boundary value problem associated to (22). Assume that p is a sufficiently regular solution on $[0, L] \times [0, T]$ of the homogeneous boundary value problem associated to (22). Then, the energy equality

$$E = \left(\int_0^L \left| \frac{\partial p}{\partial t} \right|^2 dx + c^2 \left(\int_0^L \left| \frac{\partial p}{\partial x} \right|^2 dx \right) \right)$$

is decreasing.

Moreover when $a \neq 0$ and $g(x,t) = 0$ on $[0, L] \times [0, T]$ then \mathcal{E} defined in Remark 2 is also decreasing.

If $a = 0$, let us consider once again the problem (22) where now $g(x,t) \neq 0$ on $[0, L] \times [0, T]$. Then, the energy inequality (8) is again valid where once again $E_{total}(t)$ and $E_{external\ force}$ are defined by a similar way than in Proposition 2.

Moreover when $a \neq 0$ and $g(x,t) \neq 0$ on $[0, L] \times [0, T]$ then the result of Remark 4 holds.

Proof 3 The proof of this Proposition is exactly the same as that of Proposition 1 and 2 by taking into account the remarks 2 and 4.

3.2 Third numerical explicit scheme.

By taking into account the Neumann condition at the point $x = L$, we have to calculate the approximate solution for $m + 1$ discretization points instead of m in the case of the Dirichlet boundary conditions. In agreement with section 2, the approximation of the second derivative with respect to x at the discretization points $i = 1, \dots, m$ is unchanged. For the approximation of $\frac{\partial p(L,t)}{\partial x}$, there are several methods available. In the sequel, we consider the addition of a phantom point at point number $m + 2$ and to take into account the fact that the first derivative in space at $x = L$ is null, we impose the condition $p_m = p_{m+2}$. Therefore, at point number $m + 1$ the discretization scheme is given by :

$$\frac{\partial p(L,t)}{\partial x} \approx \frac{2p_{m+1} - 2p_m}{h^2}.$$

In this case, for the discretization of the following one-dimensional Laplace equation

$$\begin{cases} \frac{d^2 q(x)}{dx^2} = f(x), \text{ on } [0, L], \\ q(0) = 0 \text{ and } \frac{\partial q(L)}{\partial x} = 0, \end{cases}$$

with mixed boundary conditions, the discretization matrix by finite difference method, denoted \mathcal{A}_{mixt} , is still positive definite and tridiagonal; the entries $a_{i,j}$ of \mathcal{A}_{mixt} are given by:

$$a_{i,j} = \begin{cases} a_{i,i} = 2.0 \text{ for } 1 \leq i \leq m + 1, \\ a_{i,i\pm 1} = -1.0 \text{ for } 1 \leq i \leq m, \\ a_{m+1,m} = -2.0, \\ a_{i,j} = 0.0 \text{ for all } (i, j) \text{ such that } |i - j| > 1. \end{cases}$$

The matrix \mathcal{A}_{mixt} is not symmetric and not normal but diagonalizable. The eigenvalues of \mathcal{A}_{mixt} are given by (see annex 7.2):

$$\lambda_l(\mathcal{A}_{mixt}) = 4 \sin^2\left(\frac{(2l-1)\pi h}{4}\right), l = 1, \dots, m + 1,$$

while the components of the associated eigenvector are given by:

$$v_i^{(l)}(\mathcal{A}_{mixt}) = \sin\left((2l-1)ih\frac{\pi}{2}\right), i, l = 1, \dots, m + 1,$$

where $m + 1$ is the size of \mathcal{A}_{mixt} . Let us denote by Λ_{mixt} the diagonal matrix constituted by the eigenvalues of \mathcal{A}_{mixt} .

Remark 12 The condition $p_m = p_{m+2}$ helps to eliminate the unknown p_{m+2} in the equation written in point number $m + 1$. We could also use the classic approximation in space:

$$\frac{\partial p(L,t)}{\partial x} \approx \frac{p_{m+1} - p_m}{h},$$

using such approximation leads to a truncation error in $O(h)$ while our choice allows to have a truncation error in $O(h^2)$.

In a similar way than in subsection 2.2, we consider first a discretization of the first derivative with respect to the time by a centered finite difference scheme. Thus, using these previous approximation, we can define an explicit scheme for the numerical solution of the damped wave equation (22). Indeed, for $n > 1$, the explicit numerical scheme is defined as follows:

$$\begin{cases} \frac{p_i^{n+1} - 2p_i^n + p_i^{n-1}}{k^2} + d\left(\frac{p_i^{n+1} - p_i^{n-1}}{2k}\right) - c^2\left(\frac{p_{i+1}^n - 2p_i^n + p_{i-1}^n}{h^2}\right) + ap_i^n = g_i^n, 1 \leq i \leq m, \\ \frac{p_{m+1}^{n+1} - 2p_{m+1}^n + p_{m+1}^{n-1}}{k^2} + d\left(\frac{p_{m+1}^{n+1} - p_{m+1}^{n-1}}{2k}\right) - 2c^2\frac{p_m^n - p_{m+1}^n}{h^2} + ap_{m+1}^n = g_{m+1}^n, \text{ for } i = m + 1, \\ p_0^n = 0 \text{ and } p_i^0 = p_0(ih), \frac{p_i^1 - p_i^0}{k} = p_1(ih), 1 \leq i \leq m + 1. \end{cases}$$

Let us multiply the previous relation by k^2 and consider also the change of variable (11); then finally, for $n > 1$, the numerical scheme can be written as follows:

$$\begin{cases} p_i^{n+1} = \frac{k^2}{\gamma+1}g_i^n + \frac{\alpha^2}{\gamma+1}(p_{i+1}^n + p_{i-1}^n) + \frac{2-2\alpha^2-\beta}{\gamma+1}p_i^n + \frac{\gamma-1}{\gamma+1}p_i^{n-1}, 1 \leq i \leq m, \\ p_{m+1}^{n+1} = \frac{k^2}{\gamma+1}g_{m+1}^n + \frac{2\alpha^2}{\gamma+1}p_m^n + \frac{2-2\alpha^2-\beta}{\gamma+1}p_{m+1}^n + \frac{\gamma-1}{\gamma+1}p_{m+1}^{n-1}, \\ p_0^n = 0 \text{ and } p_i^0 = p_0(ih), p_i^1 = p_i^0 + k.p_1(ih), 1 \leq i \leq m + 1. \end{cases} \quad (23)$$

Since $p_m = p_{m+2}$, for $n > 1$ and $1 \leq i \leq m + 1$, we can also write the previous relations as follows:

$$\begin{cases} p_i^{n+1} = \frac{k^2}{\gamma+1}g_i^n + \frac{\alpha^2}{\gamma+1}(p_{i+1}^n + p_{i-1}^n) + \frac{2-2\alpha^2-\beta}{\gamma+1}p_i^n + \frac{\gamma-1}{\gamma+1}p_i^{n-1}, \\ p_m^n = p_{m+2}^n, \\ p_0^n = 0 \text{ and } p_i^0 = p_0(ih), p_i^1 = p_i^0 + k.p_1(ih). \end{cases} \quad (24)$$

By taking into account the approximation of the derivative, according to the evaluation performed in the subsection 2.2 and by using the same notation, the truncation error is again given by:

$$F \leq Q(k^2 + h^2),$$

and we can conclude by stating the following result:

Proposition 15 *The numerical explicit scheme (24) is consistent and for $n > 1$, the numerical scheme is of order 2 in time and space.*

To study the stability of the numerical scheme by the Von Neuman method, for $n > 1$, let us write, as previously done, the homogeneous semidiscretized scheme in time. By taking the Fourier transform this leads finally to:

$$\hat{p}^{n+1}(l) = -\frac{(4\alpha^2 \sin^2(\frac{lh}{2}) - 2 + \beta)}{1 + \gamma} \hat{p}^n(l) - \frac{1 - \gamma}{1 + \gamma} \hat{p}^{n-1}(l),$$

and finally we obtain:

$$\begin{pmatrix} \hat{p}^{n+1}(l) \\ \hat{p}^n(l) \end{pmatrix} = \begin{pmatrix} -\frac{(4\alpha^2 \sin^2(\frac{lh}{2}) - 2 + \beta)}{1 + \gamma} & -\frac{1 - \gamma}{1 + \gamma} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \hat{p}^n(l) \\ \hat{p}^{n-1}(l) \end{pmatrix},$$

where

$$\mathcal{B} = \begin{pmatrix} -\frac{(4\alpha^2 \sin^2(\frac{lh}{2}) - 2 + \beta)}{1 + \gamma} & -\frac{1 - \gamma}{1 + \gamma} \\ 1 & 0 \end{pmatrix}$$

is the amplification matrix. By performing exactly the same computations, as those considered in the subsection 2.2, we finally obtain the same stability conditions:

$$\alpha \leq 1 \text{ and } \frac{\beta}{2} \leq \gamma.$$

By returning to the initial notations of problem (22), the numerical stability condition are written as follows:

$$\frac{ck}{h} \leq 1 \text{ and } ak \leq d,$$

the first one corresponding to the Courant, Friedrichs and Levy (C.F.L.) condition.

Note also that the stability condition can be obtained by using the matricial method. Hence, let consider the homogeneous numerical scheme (23) written vectorially as follows:

$$P^{n+1} = \frac{1}{\gamma + 1}(2I - \alpha^2 \mathcal{A}_{mixt} - \beta I)P^n + \frac{\gamma - 1}{\gamma + 1}P^{n-1}.$$

By writing the previous relation in the basis of the eigenvectors of \mathcal{A}_{mixt} , we obtain:

$$\bar{P}^{n+1} = \frac{1}{\gamma + 1}(2I - \alpha^2 \Lambda_{mixt} - \beta I)\bar{P}^n + \frac{\gamma - 1}{\gamma + 1}\bar{P}^{n-1},$$

which, as previously done, can be written as follows:

$$\begin{pmatrix} \bar{P}^{n+1} \\ \bar{P}^n \end{pmatrix} = \begin{pmatrix} \frac{1.0}{\gamma + 1}(2I - \alpha^2 \Lambda_{mixt} - \beta I) & \frac{\gamma - 1}{1 + \gamma}I \\ I & 0 \end{pmatrix} \begin{pmatrix} \bar{P}^n \\ \bar{P}^{n-1} \end{pmatrix},$$

which leads to the the characteristic equation:

$$\lambda^2 + \frac{(4\alpha^2 \sin^2(\frac{(2l-1)\pi h}{4})) - 2 + \beta}{1 + \gamma}\lambda - \frac{\gamma - 1}{1 + \gamma} = 0,$$

and we obtain a stability condition identical to the one obtained by Von Neumann method.

Since the numerical scheme is consistent and conditionally stable, we have a convergence result when the discretization steps in time and space tend towards to zero. Thus, we can state the following result:

Proposition 16 *The numerical scheme (24) being consistent and stable when*

$$\alpha \leq 1 \text{ and } \frac{\beta}{2} \leq \gamma,$$

is convergent when $k \rightarrow 0$ and $h \rightarrow 0$.

Therefore, the previous result expresses that stability and consistency are a necessary condition for convergence.

Remark 13 Let us express the stability conditions for the initial problem (21) with initial notations; we obtain immediately:

$$\sqrt[2]{\frac{\theta}{\mu}} \frac{k}{h} \leq 1 \text{ and } \delta k \leq \mu_1.$$

3.3 Fourth numerical explicit scheme.

For the numerical solution of problem (22), the only change with respect to subsection 3.2 relates to the discretization of the term $\frac{\partial p}{\partial t}$ where here, as in subsection 2.3, this term is discretized by a decentered scheme. Thus, for $n > 1$, the numerical scheme can be written as follows:

$$\begin{cases} \frac{p_i^{n+1} - 2p_i^n + p_i^{n-1}}{k^2} + d\left(\frac{p_i^{n+1} - p_i^n}{k}\right) - c^2\left(\frac{p_{i+1}^n - 2p_i^n + p_{i-1}^n}{h^2}\right) + ap_i^n = g_i^n, 1 \leq i \leq m, \\ \frac{p_{m+1}^{n+1} - 2p_{m+1}^n + p_{m+1}^{n-1}}{k^2} + d\left(\frac{p_{m+1}^{n+1} - p_{m+1}^n}{k_0}\right) - 2c^2\frac{p_m^n - p_{m+1}^n}{h^2} + ap_{m+1}^n = g_{m+1}^n, \\ p_0^n = 0 \text{ and } p_i^0 = p_0(ih), \frac{p_i^1 - p_i^0}{k} = p_1(ih), 1 \leq i \leq m+1. \end{cases}$$

Let us again multiply the previous relation by k^2 and consider also the change of variable (11) except for γ replaced by (13). Hence, finally, for $n > 1$, the numerical scheme can be written as follows:

$$\begin{cases} p_i^{n+1} = \frac{k^2}{\bar{\gamma}+1}g_i^n + \frac{\alpha^2}{\bar{\gamma}+1}(p_{i+1}^n + p_{i-1}^n) + \frac{\bar{\gamma}+2-2\alpha^2-\beta}{\bar{\gamma}+1}p_i^n - \frac{1}{\bar{\gamma}+1}p_i^{n-1}, 1 \leq i \leq m, \\ p_{m+1}^{n+1} = \frac{k^2}{\bar{\gamma}+1}g_{m+1}^n + \frac{2\alpha^2}{\bar{\gamma}+1}p_m^n + \frac{\bar{\gamma}+2-2\alpha^2-\beta}{\bar{\gamma}+1}p_{m+1}^n - \frac{1}{\bar{\gamma}+1}p_{m+1}^{n-1}, \\ p_0^n = 0 \text{ and } p_i^0 = p_0(ih), p_i^1 = p_i^0 + k.p_1(ih), 1 \leq i \leq m+1. \end{cases} \quad (25)$$

Since $p_m = p_{m+2}$, we can write, for $n > 1$ and $1 \leq i \leq m+1$:

$$\begin{cases} p_i^{n+1} = \frac{k^2}{\bar{\gamma}+1}g_i^n + \frac{\alpha^2}{\bar{\gamma}+1}(p_{i+1}^n + p_{i-1}^n) + \frac{\bar{\gamma}+2-2\alpha^2-\beta}{\bar{\gamma}+1}p_i^n - \frac{1}{\bar{\gamma}+1}p_i^{n-1}, \\ p_m^n = p_{m+2}^n, \\ p_0^n = 0 \text{ and } p_i^0 = p_0(ih), p_i^1 = p_i^0 + k.p_1(ih). \end{cases} \quad (26)$$

According to the result of subsection 2.3, the truncation error is given by:

$$F \leq Q(k + h^2),$$

and we can conclude by stating the following result:

Proposition 17 *The numerical explicit scheme (26) is consistent and is of order 1 in time and of order 2 in space.*

For the study of the stability of the numerical scheme using the Von Neumann method, for $n > 1$, as previously, let us write the homogeneous semidiscretized scheme. By taking the Fourier transform, this leads to:

$$\hat{p}^{n+1}(l) = \frac{(\bar{\gamma} + 2 - 4\alpha^2 \sin^2(\frac{lh}{2}) - \beta)}{1 + \bar{\gamma}} \hat{p}^n(l) - \frac{1}{1 + \bar{\gamma}} \hat{p}^{n-1}(l),$$

and finally, we obtain:

$$\begin{pmatrix} \hat{p}^{n+1}(l) \\ \hat{p}^n(l) \end{pmatrix} = \begin{pmatrix} \frac{(\bar{\gamma}+2-4\alpha^2 \sin^2(\frac{lh}{2})-\beta)}{1+\bar{\gamma}} & -\frac{1}{1+\bar{\gamma}} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \hat{p}^n(l) \\ \hat{p}^{n-1}(l) \end{pmatrix}.$$

Let

$$\mathcal{B} = \begin{pmatrix} \frac{(\bar{\gamma}+2-4\alpha^2 \sin^2(\frac{lh}{2})-\beta)}{1+\bar{\gamma}} & -\frac{1}{1+\bar{\gamma}} \\ 1 & 0 \end{pmatrix}$$

be the amplification matrix. The characteristic equation of \mathcal{B} is given by:

$$\lambda^2 - \frac{(\bar{\gamma} + 2 - 4\alpha^2 \sin^2(\frac{lh}{2}) - \beta)}{1 + \bar{\gamma}} \lambda + \frac{1}{1 + \bar{\gamma}} = 0.$$

The product of the roots is equal to $\frac{1}{1+\bar{\gamma}} < 1$. Thus, the roots have the same sign, the sum of the roots is equal to $\frac{(\bar{\gamma}+2-4\alpha^2 \sin^2(\frac{lh}{2})-\beta)}{1+\bar{\gamma}}$. We have to verify the two inequalities:

$$-2 \leq \frac{(\bar{\gamma} + 2 - 4\alpha^2 \sin^2(\frac{lh}{2}) - \beta)}{1 + \bar{\gamma}} \leq 2.$$

The inequality from the right leads to $-4\alpha^2 - \beta \leq 0$, which is always true and the inequality from the left leads to:

$$4\alpha^2 \sin^2(\frac{lh}{2}) + \beta \leq 4 + 3\bar{\gamma},$$

which leads to

$$\alpha^2 \leq 1 \text{ and } \beta \leq 3\bar{\gamma},$$

since obviously $\sin^2(\frac{lh}{2}) \leq 1$. By returning to the initial notations of the problem (22), the numerical stability of the numerical scheme is obtained when:

$$\frac{ck}{h} \leq 1 \text{ and } \frac{\beta}{3} = \frac{ak^2}{3} \leq \bar{\gamma} = dk,$$

which, by using the initial notations defined in problem (21), leads to:

$$\sqrt[2]{\frac{\theta}{\mu}} \frac{k}{h} \leq 1 \text{ and } \frac{\delta k}{3} \leq \mu_1.$$

Remark 14 *Once again, note that for the numerical scheme (26), we can consider a time step k three times larger than the one to take for the numerical scheme (24). Therefore, it is very much appreciated; however, the precision is less good since the truncation error is in $O(k)$.*

Let us find the stability condition using the matricial method. We can write the homogeneous scheme vectorially as follows:

$$P^{n+1} = \frac{1}{\bar{\gamma} + 1} ((2 + \bar{\gamma})I - \alpha^2 \mathcal{A}_{mixt} - \beta I) P^n - \frac{1}{\bar{\gamma} + 1} P^{n-1}.$$

In the basis of the eigenvectors of \mathcal{A}_{mixt} , we obtain the following relation:

$$\bar{P}^{n+1} = \frac{1}{\bar{\gamma} + 1} ((2 + \bar{\gamma})I - \alpha^2 \Lambda_{mixt} - \beta I) \bar{P}^n - \frac{1}{\bar{\gamma} + 1} \bar{P}^{n-1},$$

which can also be written as follows:

$$\begin{pmatrix} \bar{P}^{n+1} \\ \bar{P}^n \end{pmatrix} = \begin{pmatrix} \frac{1}{\bar{\gamma}+1}((2+\bar{\gamma})I - \alpha^2 \Lambda_{mixt} - \beta I) - \frac{1}{1+\bar{\gamma}}I \\ I \end{pmatrix} \begin{pmatrix} \bar{P}^n \\ \bar{P}^{n-1} \end{pmatrix},$$

where

$$\mathcal{B} = \begin{pmatrix} \frac{1}{\bar{\gamma}+1}((2+\bar{\gamma})I - \alpha^2 \Lambda_{mixt} - \beta I) - \frac{1}{1+\bar{\gamma}}I \\ I \end{pmatrix}$$

is the amplification matrix. Thus, the characteristic equation of \mathcal{B} is given by:

$$\lambda^2 - \frac{(\bar{\gamma} + 2 - 4\alpha^2 \sin^2(\frac{(2l-1)\pi h}{4}) - \beta)}{1 + \bar{\gamma}} \lambda + \frac{1}{1 + \bar{\gamma}} = 0,$$

and similarly to the Von Neumann method, we find the following stability conditions:

$$\alpha \leq 1 \text{ and } \beta \leq 3\bar{\gamma}.$$

Concerning the convergence, we can conclude similarly than previously stated:

Proposition 18 *The numerical scheme (26) being consistent and stable when $\alpha \leq 1$ and $\beta \leq 3\bar{\gamma}$ then it is convergent when $k \rightarrow 0$ and $h \rightarrow 0$.*

3.4 Fourth numerical implicit scheme.

In a similar way to the approach considered in the subsection 2.4, let us construct as follows an implicit scheme, in the case where the first derivative with respect to the time is discretized by a centered scheme. Then, for $n > 1$, we obtain:

$$\begin{cases} \frac{p_i^{n+1} - 2p_i^n + p_i^{n-1}}{k^2} + d \left(\frac{p_i^{n+1} - p_i^{n-1}}{2k} \right) - c^2 \left(\frac{p_{i+1}^{n+1} - 2p_i^{n+1} + p_{i-1}^{n+1}}{h^2} \right) + ap_i^{n+1} = g_i^{n+1}, 1 \leq i \leq m, \\ \frac{p_{m+1}^{n+1} - 2p_{m+1}^n + p_{m+1}^{n-1}}{k^2} + d \left(\frac{p_{m+1}^{n+1} - p_{m+1}^{n-1}}{2k} \right) - 2c^2 \frac{p_m^{n+1} - p_{m+1}^{n+1}}{h^2} + ap_{m+1}^{n+1} = g_{m+1}^n, i = m + 1, \\ p_0^{n+1} = 0 \text{ and } p_i^0 = p_0(ih), \frac{p_i^1 - p_i^0}{k} = p_1(ih), 1 \leq i \leq m + 1. \end{cases}$$

Let us again multiply the previous relation by k^2 and consider also the change of variable (11); then, finally since $p_m = p_{m+2}$, the implicit numerical scheme can be written as follows for $n > 1$ and $1 \leq i \leq m + 1$:

$$\begin{cases} -\alpha^2 p_{i+1}^{n+1} + (\gamma + 1 + \beta + 2\alpha^2) p_i^{n+1} - \alpha^2 p_{i-1}^{n+1} = k^2 g_i^{n+1} + 2p_i^n + (\gamma - 1) p_i^{n-1}, \\ p_m^{n+1} = p_{m+2}^{n+1}, \\ p_0^{n+1} = 0 \text{ and } p_i^0 = p_0(ih), \frac{p_i^1 - p_i^0}{k} = p_1(ih), \end{cases} \quad (27)$$

and thus, P^{n+1} is solution of the linear system:

$$((\gamma + \beta + 1)I + \alpha^2 \mathcal{A}_{mixt})P^{n+1} = k^2 G^{n+1} + 2P^n + (\gamma - 1)P^{n-1},$$

of size $(m + 1)$. Since the matrix $((\gamma + \beta + 1)I + \alpha^2 \mathcal{A}_{mixt})$ is strictly diagonal dominant, consequently it is invertible and we can solve the linear system by the TDMA method.

Note also that the truncation error is given by:

$$F \leq Q(k^2 + h^2),$$

and the implicit numerical scheme (27) is consistent.

Proposition 19 *The implicit numerical scheme (27) is consistent and of order 2 in time and in space.*

Let us now study the stability of the numerical implicit scheme (27) by the Von Neumann method. Thus, let us consider the homogeneous scheme semidiscretized with respect to the time; we then obtain for $n > 1$:

$$-\alpha^2 p^{n+1}(x+h) + (\gamma + 1 + \beta + 2\alpha^2)p^{n+1}(x) - \alpha^2 p^{n+1}(x-h) = 2p^n(x) + (\gamma - 1)p^{n-1}(x),$$

which, after performing the Fourier transform, leads to:

$$\hat{p}^{n+1}(l) = \frac{2}{(4\alpha^2 \sin^2(\frac{lh}{2}) + \gamma + \beta + 1)} \hat{p}^n(l) + \frac{\gamma - 1}{(4\alpha^2 \sin^2(\frac{lh}{2}) + \gamma + \beta + 1)} \hat{p}^{n-1}(l),$$

and the amplification matrix is defined as follows:

$$\mathcal{B} = \begin{pmatrix} \frac{2}{(4\alpha^2 \sin^2(\frac{lh}{2}) + \gamma + \beta + 1)} & \frac{\gamma - 1}{(4\alpha^2 \sin^2(\frac{lh}{2}) + \gamma + \beta + 1)} \\ 1 & 0 \end{pmatrix},$$

and the characteristic equation is given by:

$$\lambda^2 - \frac{2}{(4\alpha^2 \sin^2(\frac{lh}{2}) + \gamma + \beta + 1)} \lambda - \frac{\gamma - 1}{(4\alpha^2 \sin^2(\frac{lh}{2}) + \gamma + \beta + 1)} = 0.$$

We are therefore in a situation similar to the one encountered in subsection 2.4 with the same sum and product of eigenvalues for \mathcal{B} which allows us to conclude on the unconditional stability of the implicit numerical scheme.

Proposition 20 *The numerical implicit scheme (27) is unconditional stable.*

As above, this property of unconditional stability can be obtained by the matricial method. In the eigenvector basis, the homogeneous system is written as follows:

$$\bar{P}^{n+1} = ((\gamma + \beta + 1)I + \alpha^2 \Lambda_{mixt})^{-1} \cdot (2\bar{P}^n + (\gamma - 1)\bar{P}^{n-1}),$$

and the amplification matrix is given by:

$$\mathcal{B} = \begin{pmatrix} \frac{2}{(4\alpha^2 \sin^2(\frac{(2l-1)\pi h}{4}) + \gamma + \beta + 1)} & \frac{\gamma - 1}{(4\alpha^2 \sin^2(\frac{(2l-1)\pi h}{4}) + \gamma + \beta + 1)} \\ 1 & 0 \end{pmatrix}.$$

As it was stated before, the eigenvalues of this matrix are of modulus strictly less than one, allowing us to conclude, once again, that the implicit numerical scheme is unconditionally stable. In addition, we have the following result:

Proposition 21 *The numerical implicit scheme (27) being consistent and unconditionally stable is also convergent when $k \rightarrow 0$ and $h \rightarrow 0$.*

3.5 Fifth numerical implicit scheme.

The construction of this new scheme is similar to the one considered for the scheme (27) except that the first derivative in time is discretized by a decentered scheme. For $n > 1$, the implicit numerical scheme is defined as follows:

$$\begin{cases} \frac{p_i^{n+1} - 2p_i^n + p_i^{n-1}}{k^2} + d\left(\frac{p_i^{n+1} - p_i^n}{k}\right) - c^2\left(\frac{p_{i+1}^{n+1} - 2p_i^{n+1} + p_{i-1}^{n+1}}{h^2}\right) + ap_i^{n+1} = g_i^{n+1}, 1 \leq i \leq m, \\ \frac{p_{m+1}^{n+1} - 2p_{m+1}^n + p_{m+1}^{n-1}}{k^2} + d\left(\frac{p_{m+1}^{n+1} - p_{m+1}^n}{k}\right) - 2c^2\frac{p_m^{n+1} - p_{m+1}^{n+1}}{h^2} + ap_{m+1}^{n+1} = g_{m+1}^{n+1}, \\ p_0^{n+1} = 0 \text{ and } p_i^0 = p_0(ih), \frac{p_i^1 - p_i^0}{k} = p_1(ih), 1 \leq i \leq m+1. \end{cases}$$

In a similar way than before, by multiplying by k^2 and by taking into account the previous change of variables (11) and (13), for $n > 1$ and $1 \leq i \leq m+1$, the final implicit numerical scheme can be written as follows:

$$\begin{cases} -\alpha^2 p_{i+1}^{n+1} + (\bar{\gamma} + 1 + \beta + 2\alpha^2) p_i^{n+1} - \alpha^2 p_{i-1}^{n+1} = k^2 g_i^{n+1} + (2 + \bar{\gamma}) p_i^n - p_i^{n-1}, \\ p_m^{n+1} = p_{m+2}^{n+1}, \\ p_0^{n+1} = 0 \text{ and } p_i^0 = p_0(ih), \frac{p_i^1 - p_i^0}{k} = p_1(ih), \end{cases} \quad (28)$$

and P^{n+1} is solution of the linear system of size $(m+1)$:

$$((\bar{\gamma} + \beta + 1)I + \alpha^2 \mathcal{A}_{mixt}) P^{n+1} = k^2 G^{n+1} + (2 + \bar{\gamma}) P^n - P^{n-1}.$$

Since the matrix $((\bar{\gamma} + \beta + 1)I + \alpha^2 \mathcal{A}_{mixt})$ is strictly diagonal dominant, consequently, it is invertible and we can solve the linear system by the TDMA method.

Note also that the truncation error is given by:

$$F \leq Q(k + h^2),$$

and we can conclude by providing the following Proposition.

Proposition 22 *The implicit numerical scheme (28) is consistent and of order 1 in time and of order 2 in space.*

Similarly than in subsection 2.5, let us study first the stability of scheme (28) by using the Von Neumann method. Hence, considering, once again, the homogeneous semidiscretized scheme in time and by taking the Fourier transform of the resulting writing, leads to:

$$\hat{p}^{n+1}(l) = \frac{2 + \bar{\gamma}}{(4\alpha^2 \sin^2(\frac{l,h}{2}) + \bar{\gamma} + \beta + 1)} \hat{p}^n(l) - \frac{1}{(4\alpha^2 \sin^2(\frac{l,h}{2}) + \bar{\gamma} + \beta + 1)} \hat{p}^{n-1}(l).$$

In a similar way than the one in subsection 2.5, we obtain the following amplification matrix:

$$\mathcal{B} = \begin{pmatrix} \frac{2+\bar{\gamma}}{4\alpha^2 \sin^2(\frac{Lh}{2})+\bar{\gamma}+\beta+1} & -\frac{1}{4\alpha^2 \sin^2(\frac{Lh}{2})+\bar{\gamma}+\beta+1} \\ 1 & 0 \end{pmatrix},$$

where the eigenvalues of \mathcal{B} satisfy the characteristic equation:

$$\lambda^2 - \frac{2+\bar{\gamma}}{4\alpha^2 \sin^2(\frac{Lh}{2})+\bar{\gamma}+\beta+1} \lambda + \frac{1}{4\alpha^2 \sin^2(\frac{Lh}{2})+\bar{\gamma}+\beta+1} = 0.$$

Thus, by a reasoning analogous to the one considered in subsection 2.5, we obtain another unconditional stability property.

Proposition 23 *The implicit numerical scheme (28) is unconditionally stable.*

Once again, we can find this property by using the matricial method. Indeed by writing the homogeneous scheme in the basis of eigenvectors this leads to:

$$\bar{P}^{n+1} = ((\bar{\gamma} + \beta + 1)I + \alpha^2 \Lambda_{mixt})^{-1}((2 + \bar{\gamma})\bar{P}^n - \bar{P}^{n-1}),$$

and the amplification matrix is given by:

$$\mathcal{B} = \begin{pmatrix} \frac{2+\bar{\gamma}}{4\alpha^2 \sin^2(\frac{(2l-1)\pi h}{4})+\bar{\gamma}+\beta+1} & -\frac{1}{4\alpha^2 \sin^2(\frac{(2l-1)\pi h}{4})+\bar{\gamma}+\beta+1} \\ 1 & 0 \end{pmatrix},$$

with a characteristic equation given by:

$$\lambda^2 - \frac{2+\bar{\gamma}}{4\alpha^2 \sin^2(\frac{(2l-1)\pi h}{4})+\bar{\gamma}+\beta+1} \lambda + \frac{1}{4\alpha^2 \sin^2(\frac{(2l-1)\pi h}{4})+\bar{\gamma}+\beta+1} = 0.$$

As previously seen, we check that the eigenvalues of \mathcal{B} are of modulus less than one.

Once again the consistence and the unconditional stability give rise to the following result:

Proposition 24 *The numerical implicit scheme (28) being consistent and unconditionally stable, is also convergent when $k \rightarrow 0$ and $h \rightarrow 0$.*

3.6 Numerical implicit scheme derived from the Gear scheme.

As previously seen in subsection 2.6 let us approximate the first derivative with respect to the time, by the following Gear scheme of order two with respect to the time. This yields:

$$\frac{3p_i^{n+1} - 4p_i^n + p_i^{n-1}}{2k} = \frac{\partial p(ih, (n+1)k)}{\partial t} - \frac{k^2}{3} \frac{\partial^3 p(ih, (n+1)k)}{\partial t^3} + O(k^2).$$

Then, by combining this previous approximation of the first derivative with respect to the time with the previous numerical scheme for the discretization of equation (22), for $n > 1$, this leads to:

$$\begin{cases} \frac{p_i^{n+1}-2p_i^n+p_i^{n-1}}{k^2} + d\left(\frac{3p_i^{n+1}-4p_i^n+p_i^{n-1}}{2k}\right) - c^2\left(\frac{p_{i+1}^{n+1}-2p_i^{n+1}+p_{i-1}^{n+1}}{h^2}\right) + ap_i^{n+1} = g_i^{n+1}, 1 \leq i \leq m, \\ \frac{p_{m+1}^{n+1}-2p_{m+1}^n+p_{m+1}^{n-1}}{k^2} + d\left(\frac{3p_{m+1}^{n+1}-4p_{m+1}^n+p_{m+1}^{n-1}}{2k}\right) - 2c^2\left(\frac{p_m^{n+1}-p_{m+1}^{n+1}}{h^2}\right) + ap_{m+1}^{n+1} = g_{m+1}^{n+1}, i = m+1, \\ p_0^n = 0 \text{ and } p_i^0 = p_0(ih), \frac{p_i^1-p_i^0}{k} = p_1(ih), 1 \leq i \leq m+1. \end{cases}$$

and since $p_m = p_{m+2}$, we obtain finally, for $n > 1$ and $1 \leq i \leq m+1$:

$$\begin{cases} \frac{p_i^{n+1}-2p_i^n+p_i^{n-1}}{k^2} + d\left(\frac{3p_i^{n+1}-4p_i^n+p_i^{n-1}}{2k}\right) - c^2\left(\frac{p_{i+1}^{n+1}-2p_i^{n+1}+p_{i-1}^{n+1}}{h^2}\right) + ap_i^{n+1} = g_i^{m+1}, \\ p_m^{n+1} = p_{m+2}^{n+1}, \\ p_0^n = p_{m+1}^n = 0 \text{ and } p_i^0 = p_0(ih), \frac{p_i^1-p_i^0}{k} = p_1(ih), 1 \leq i \leq m. \end{cases} \quad (29)$$

The Gear scheme being of order two with respect to the time then, due to the result presented in subsection 2.6 concerning the order of approximation of the second derivative with respect to the time and to the space, the truncation error verifies once again:

$$F \leq Q(k^2 + h^2);$$

thus, this new scheme derived from the Gear scheme is of order two in space and in time, so that we can conclude by stating the following result:

Proposition 25 *The numerical explicit scheme (29) is consistent.*

Let us multiply by k^2 and by using the change of variable (11), the numerical scheme can be written as follows, for $n > 1$ and $1 \leq i \leq m+1$:

$$\begin{cases} -\alpha^2 p_{i+1}^{n+1} + (3\gamma + \beta + 2\alpha^2 + 1)p_i^{n+1} - \alpha^2 p_{i-1}^{n+1} = k^2 g_i^{n+1} + 2(1 + 2\gamma)p_i^n - (1 + \gamma)p_i^{n-1}, \\ p_m^{n+1} = p_{m+2}^{n+1}, \\ p_0^n = 0 \text{ and } p_i^0 = p_0(ih), \frac{p_i^1-p_i^0}{k} = p_1(ih). \end{cases} \quad (30)$$

This relation can be written vectorially as follows:

$$((3\gamma + \beta + 1)I + \alpha^2 \mathcal{A}_{mixt})P^{n+1} = k^2 G^{n+1} + 2(1 + 2\gamma)P^n - (1 + \gamma)P^{n-1}.$$

Consider now the associated homogeneous semidiscretized scheme in time and let us perform the Fourier transform. This leads to the following relation:

$$(4\alpha^2 \sin^2(\frac{lh}{2}) + 3\gamma + \beta + 1)\hat{p}^{n+1} = 2(1 + 2\gamma)\hat{p}^n - (1 + \gamma)\hat{p}^{n-1},$$

or also

$$\hat{p}^{n+1} = \frac{2(1 + 2\gamma)}{4\alpha^2 \sin^2(\frac{lh}{2}) + 3\gamma + \beta + 1} \hat{p}^n - \frac{1 + \gamma}{4\alpha^2 \sin^2(\frac{lh}{2}) + 3\gamma + \beta + 1} \hat{p}^{n-1}.$$

This relation can also be written in the following form:

$$\begin{pmatrix} \hat{p}^{n+1}(l) \\ \hat{p}^n(l) \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2(1+2\gamma)}{4\alpha^2 \sin^2(\frac{lh}{2}) + 3\gamma + \beta + 1} & -\frac{1+\gamma}{4\alpha^2 \sin^2(\frac{lh}{2}) + 3\gamma + \beta + 1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \hat{p}^n(l) \\ \hat{p}^{n-1}(l) \end{pmatrix}.$$

Consequently, the eigenvalues of the amplification matrix satisfy the following second degree equation:

$$\lambda^2 - \frac{2(1+2\gamma)}{4\alpha^2 \sin^2(\frac{lh}{2}) + 3\gamma + \beta + 1} \lambda + \frac{1+\gamma}{4\alpha^2 \sin^2(\frac{lh}{2}) + 3\gamma + \beta + 1} = 0.$$

The product of the root is positive, so that the eigenvalues have the same sign. In addition, the sum of the roots is positive, and the eigenvalues are positive. Thus the implicit numerical scheme is stable when

$$-2 \leq \frac{2(1+2\gamma)}{4\alpha^2 \sin^2(\frac{lh}{2}) + 3\gamma + \beta + 1} \leq 2.$$

We can easily see that these inequalities are always verified and we can conclude that the implicit scheme derived from the Gear scheme (30) is unconditionally stable. Moreover, since this scheme is consistent, then it is also convergent when $h \rightarrow 0$ and $k \rightarrow 0$.

Once again, we can obtain the same result of unconditional stability by using the matricial method and state the same conclusion of convergence of the numerical implicit scheme when $h \rightarrow 0$ and $k \rightarrow 0$.

Remark 15 *In a similar way to the study presented in the subsection 2.7, we can also draw up numerical schemes when the coefficients of the damped wave equation are no longer constant. As in the subsection 2.7, we establish identical conditional numerical stability conditions for the third and fourth explicit schemes and an unconditional stability property for the fourth, the fifth implicit schemes and the implicit numerical scheme derived from the Gear scheme.*

4 Numerical schemes for the solution of the multidimensional damped wave equation.

Let us now consider the multidimensional damped wave equation equipped with only homogeneous Dirichlet boundary conditions in a bounded domain Ω ; let $\partial\Omega$ be the boundary of Ω . In what follows in the bi-dimensional case $\Omega = [0, L] \times [0, L]$ while in the three-dimensional case $\Omega = [0, L]^3$. Hence, in all cases the problem to solve numerically is formulated as follows:

$$\begin{cases} \mu \frac{\partial^2 p(\mathbf{x}, \mathbf{t})}{\partial t^2} - \theta \Delta p(\mathbf{x}, \mathbf{t}) + \mu_1 \frac{\partial p(\mathbf{x}, \mathbf{t})}{\partial t} + \delta p(\mathbf{x}, \mathbf{t}) = \bar{\mathbf{g}}(\mathbf{x}, \mathbf{t}), & \text{on } \bar{\Omega} \times [0, \mathbf{T}], \\ p(\mathbf{x}, \mathbf{t}) = \mathbf{0} & \text{on } \partial\Omega \text{ for } \mathbf{t} \geq 0, \\ p(\mathbf{x}, \mathbf{0}) = \mathbf{p}_0(\mathbf{x}) \text{ and } \frac{\partial p(\mathbf{x}, \mathbf{0})}{\partial t} = \mathbf{p}_1(\mathbf{x}) & \text{for } \mathbf{x} \in \Omega, \end{cases} \quad (31)$$

where in the bi-dimensional case $\mathbf{x} = (x, y)$ while in the three-dimensional case $\mathbf{x} = (x, y, z)$. Consider the change of variable (4) so that the previous problem is written as follows:

$$\begin{cases} \frac{\partial^2 p(\mathbf{x}, \mathbf{t})}{\partial t^2} - c^2 \Delta p(\mathbf{x}, \mathbf{t}) + \mathbf{d} \frac{\partial p(\mathbf{x}, \mathbf{t})}{\partial \mathbf{t}} + \mathbf{a} p(\mathbf{x}, \mathbf{t}) = \mathbf{g}(\mathbf{x}, \mathbf{t}), & \text{on } \bar{\Omega} \times [0, \mathbf{T}], \\ p(\mathbf{x}, \mathbf{t}) = \mathbf{0} & \text{on } \partial\Omega \text{ for } \mathbf{t} \geq \mathbf{0}, \\ p(\mathbf{x}, \mathbf{0}) = \mathbf{p}_0(\mathbf{x}) & \text{and } \frac{\partial p(\mathbf{x}, \mathbf{0})}{\partial \mathbf{t}} = \mathbf{p}_1(\mathbf{x}) \text{ for } \mathbf{x} \in \Omega. \end{cases} \quad (32)$$

Remark 16 *In the present section, we restrict ourselves to the case of Dirichlet boundary conditions, since if we consider the case of mixed boundary conditions, we would have to consider situations where more than one part of the boundary is subject to a Neumann condition; such a situation would considerably complicate the presentation of the paper. In such cases, the stability of the schemes can be analyzed by computing by a numerical way the eigenvalues of the discretization matrix in space and then, by using these values to check the stability conditions. However, we refer to section 7 for the determination of the eigenvalues of the discretization matrices considered here in simple geometric shapes.*

4.1 Energy inequality.

First of all, let us extend the results of the subsection 2.1 to the two-dimensional and three-dimensional cases.

Proposition 26 *For (32) assume that $p(\mathbf{x}, \mathbf{t})$ is sufficiently regular on $\Omega \times [0, T]$. Then let*

$$E_{total}(t) = \int_{\Omega} \left| \frac{\partial p(\mathbf{x}, \mathbf{t})}{\partial t} \right|^2 d\mathbf{x} + c^2 \int_{\Omega} |\nabla(p(\mathbf{x}, \mathbf{t}))|^2 d\mathbf{x} + a \int_{\Omega} p^2(\mathbf{x}, \mathbf{t}) d\mathbf{x},$$

and

$$E_{externalforce} = \int_0^t \int_{\Omega} g^2(\mathbf{x}, s) d\mathbf{x} ds;$$

then, we have,

$$E_{total}(t) \leq (E_{total}(0) + E_{externalforce}) \cdot e^t.$$

Consequently, we obtain a continuous dependence on initial conditions and supply of energy. In addition when (32) is homogeneous the energy E_{total} is decreasing.

Proof 4 *Assume that $p(\mathbf{x}, \mathbf{t})$ is sufficiently regular and let us multiply (32) by $\frac{\partial p}{\partial t}$ and integrate on the domain Ω . Since the function p is regular, by exchanging the operators for integration in space and derivation in time, we obtain:*

$$\frac{d}{dt} \left(\int_{\Omega} \left| \frac{\partial p}{\partial t} \right|^2 d\mathbf{x} \right) + c^2 \frac{d}{dt} \left(\int_{\Omega} |\nabla(p)|^2 d\mathbf{x} \right) + a \frac{d}{dt} \left(\int_{\Omega} p^2 d\mathbf{x} \right) \leq 2 \int_{\Omega} g \frac{\partial p}{\partial t} d\mathbf{x}.$$

By integrating with respect to the time, we get:

$$\begin{aligned} & \int_{\Omega} \left| \frac{\partial p(\mathbf{x}, t)}{\partial t} \right|^2 d\mathbf{x} + c^2 \int_{\Omega} |\nabla(p(\mathbf{x}, t))|^2 d\mathbf{x} + a \int_{\Omega} p^2(\mathbf{x}, t) d\mathbf{x} \\ & \leq \int_{\Omega} p_1^2(\mathbf{x}) d\mathbf{x} + c^2 \int_{\Omega} |\nabla(p_0(\mathbf{x}))|^2 d\mathbf{x} + a \int_{\Omega} p_0^2(\mathbf{x}) d\mathbf{x} + 2 \int_0^t \int_{\Omega} g(\mathbf{x}, s) \frac{\partial p(\mathbf{x}, s)}{\partial t} d\mathbf{x} ds. \end{aligned}$$

Let us consider now the term:

$$\int_{\Omega} g(\mathbf{x}, s) \frac{\partial p(\mathbf{x}, s)}{\partial t} d\mathbf{x} ds;$$

according to Schwarz inequality, and then classically maximizing, we have:

$$\begin{aligned} \int_0^t \int_{\Omega} g(\mathbf{x}, s) \frac{\partial p(\mathbf{x}, s)}{\partial t} d\mathbf{x} ds &\leq \left(\int_0^t \int_{\Omega} g^2(\mathbf{x}, s) d\mathbf{x} ds \right)^{\frac{1}{2}} \left(\int_0^t \int_{\Omega} \left| \frac{\partial p(\mathbf{x}, s)}{\partial t} \right|^2 d\mathbf{x} ds \right)^{\frac{1}{2}}, \\ &\leq \frac{1}{2} \left(\int_0^t \int_{\Omega} g^2(\mathbf{x}, s) d\mathbf{x} ds + \int_0^t \int_{\Omega} \left| \frac{\partial p(\mathbf{x}, s)}{\partial t} \right|^2 d\mathbf{x} ds \right). \end{aligned}$$

Finally, applying the above majoration, this leads to:

$$\begin{aligned} &\int_{\Omega} \left| \frac{\partial p(\mathbf{x}, t)}{\partial t} \right|^2 d\mathbf{x} + c^2 \int_{\Omega} |\nabla(p(\mathbf{x}, t))|^2 d\mathbf{x} + a \int_{\Omega} p^2(\mathbf{x}, t) d\mathbf{x} \\ &\leq \int_{\Omega} p_1^2(\mathbf{x}) d\mathbf{x} + c^2 \int_{\Omega} |\nabla(p_0(\mathbf{x}))|^2 d\mathbf{x} + a \int_{\Omega} p_0^2(\mathbf{x}) d\mathbf{x} + \int_0^t \int_{\Omega} \mathbf{g}^2(\mathbf{x}, s) d\mathbf{x} ds \\ &+ \int_0^t \int_{\Omega} \left| \frac{\partial p(\mathbf{x}, s)}{\partial t} \right|^2 d\mathbf{x} ds. \end{aligned}$$

Now, by applying the Gronwall lemma, this leads to:

$$\begin{aligned} &\int_{\Omega} \left| \frac{\partial p(\mathbf{x}, t)}{\partial t} \right|^2 d\mathbf{x} + c^2 \int_{\Omega} |\nabla(p(\mathbf{x}, t))|^2 d\mathbf{x} + a \int_{\Omega} p^2(\mathbf{x}, t) d\mathbf{x} \\ &\leq \left(\int_{\Omega} p_1^2(\mathbf{x}) d\mathbf{x} + c^2 \int_{\Omega} |\nabla(p_0(\mathbf{x}))|^2 d\mathbf{x} + a \int_{\Omega} p_0^2(\mathbf{x}) d\mathbf{x} + \int_0^t \int_{\Omega} \mathbf{g}^2(\mathbf{x}, s) d\mathbf{x} ds \right) e^t. \end{aligned}$$

In addition, in the case where $g(\mathbf{x}, t) \equiv 0$ obviously, the total energy is decreasing and the proof is complete.

4.2 Fifth explicite scheme.

In what follows, we will consider both the two-dimensional and the three-dimensional cases when the first derivative is discretized by a centered scheme. In the sequel, we will also use the notations of subsection 2.2.

4.2.1 Two-dimensional damped wave equation.

In this case, for $1 \leq i, j \leq m$ and $n > 1$, the numerical scheme is defined by:

$$\begin{cases} \frac{p_{i,j}^{n+1} - 2p_{i,j}^n + p_{i,j}^{n-1}}{k^2} + d \left(\frac{p_{i,j}^{n+1} - p_{i,j}^{n-1}}{2k} \right) - c^2 \left(\frac{p_{i,j+1}^n + p_{i+1,j}^n - 4p_{i,j}^n + p_{i-1,j}^n + p_{i,j-1}^n}{h^2} \right) + ap_{i,j}^n = g_{i,j}^n, \\ p_{0,j}^n = p_{m+1,j}^n = p_{i,0}^n = p_{i,m+1}^n = 0 \text{ and } p_{i,j}^0 = p_0(ih, jh), \frac{p_{i,j}^1 - p_{i,j}^0}{k} = p_1(ih, jh). \end{cases}$$

Let us multiply the previous relation by k^2 and consider also the change of variable (11), such that the previous numerical scheme can be written as follows:

$$(\gamma+1)p_{i,j}^{n+1} = k^2 g_{i,j}^n + \alpha^2 (p_{i+1,j}^n + p_{i-1,j}^n + p_{i,j+1}^n + p_{i,j-1}^n) + (2-4\alpha^2-\beta)p_{i,j}^n + (\gamma-1)p_{i,j}^{n-1},$$

or finally, for $1 \leq i, j \leq m$ and $n > 1$, as:

$$\begin{cases} p_{i,j}^{n+1} = \frac{k^2}{\gamma+1} g_{i,j}^n + \frac{\alpha^2}{\gamma+1} (p_{i+1,j}^n + p_{i-1,j}^n + p_{i,j+1}^n + p_{i,j-1}^n) + \frac{2-4\alpha^2-\beta}{\gamma+1} p_{i,j}^n + \frac{\gamma-1}{\gamma+1} p_{i,j}^{n-1}, \\ p_{0,j}^n = p_{m+1,j}^n = p_{i,0}^n = p_{i,m+1}^n = 0 \text{ and } p_{i,j}^0 = p_0(ih, jh), p_{i,j}^1 = p_{i,j}^0 + kp_1(ih, jh). \end{cases} \quad (33)$$

This relation can be written vectorially, as follows:

$$P^{n+1} = \frac{k^2}{\gamma+1} G^n + \frac{1}{\gamma+1} ((2-\beta)I - \alpha^2 \mathcal{A}_{2D}) P^n + \frac{\gamma-1}{\gamma+1} P^{n-1}, \text{ for } n > 1, \quad (34)$$

where \mathcal{A}_{2D} is the discretization matrix of the Laplacian.

By concerning the truncation error, we obtain the same result than the one obtained in subsection 2.2, i.e.

$$F \leq Q(h^2 + k^2).$$

Thus, this yields the following Proposition:

Proposition 27 *The explicit numerical scheme (33) is consistent and for $n > 1$ and it is of order 2 in space and in time.*

For the study of stability, we limit ourselves to the use of the matricial method. Indeed, the eigenvalues of the matrix \mathcal{A}_{2D} are given by (see annex 7.3):

$$\lambda_{k,l}(\mathcal{A}_{2D}) = 4(\sin^2(\frac{k\pi h}{2}) + \sin^2(\frac{l\pi h}{2})), 1 \leq k, l \leq m,$$

and the associated components of the eigenvectors are given by:

$$v_{i,j}^{k,l} = \sin(ik\pi h) \sin(jl\pi h), 1 \leq i, j, k, l \leq m.$$

Thus expressed in the basis of the eigenvectors of \mathcal{A}_{2D} , the homogeneous relation (34) is written as follows:

$$\bar{P}^{n+1} = \frac{1}{\gamma+1} ((2-\beta)I - \alpha^2 \Lambda_{2D}) \bar{P}^n + \frac{\gamma-1}{\gamma+1} \bar{P}^{n-1}.$$

Similarly to the previous sections, we obtain finally the following relation:

$$\begin{pmatrix} \bar{P}^{n+1} \\ \bar{P}^n \end{pmatrix} = \begin{pmatrix} \frac{1-\gamma}{\gamma+1} ((2-\beta)I - \alpha^2 \Lambda_{2D}) & \frac{\gamma-1}{1+\gamma} I \\ I & 0 \end{pmatrix} \begin{pmatrix} \bar{P}^n \\ \bar{P}^{n-1} \end{pmatrix},$$

and the eigenvalues λ of the amplification matrix thus satisfy the second degree equation:

$$\lambda^2 + \frac{(4\alpha^2(\sin^2(\frac{k\pi h}{2}) + \sin^2(\frac{l\pi h}{2})) - 2 + \beta)}{1 + \gamma} \lambda + \frac{1 - \gamma}{1 + \gamma} = 0.$$

By a similar way than in subsection 2.2, the stability condition is obtained when the sum of the roots is less than two, i.e.

$$-2 \leq -\frac{(4\alpha^2(\sin^2(\frac{k\pi h}{2}) + \sin^2(\frac{l\pi h}{2})) - 2 + \beta)}{1 + \gamma} \leq 2.$$

This leads to the following result:

Proposition 28 *The explicit numerical scheme (33) being consistent and stable when*

$$\alpha \leq \frac{\sqrt[3]{2}}{2} \text{ and } \frac{\beta}{2} \leq \gamma,$$

is also convergent when $h \rightarrow 0$ and $k \rightarrow 0$.

4.2.2 Three-dimensional damped wave equation.

In the three-dimensional case, the approach is similar than in the previous subsection. Since the consistency of the numerical scheme is obviously still verified then the truncation error is again given by:

$$F \leq Q(h^2 + k^2).$$

Thus, we will simply indicate the formulation of the numerical scheme and we will establish the stability conditions.

The numerical explicit scheme can be defined vectorially as:

$$P^{n+1} = \frac{k^2}{\gamma + 1} G^n + \frac{1}{\gamma + 1} ((2 - \beta)I - \alpha^2 \mathcal{A}_{3D}) P^n + \frac{\gamma - 1}{\gamma + 1} P^{n-1}, \quad (35)$$

where \mathcal{A}_{3D} is the discretization matrix of the Laplacian. If we detail and keep the same notations, for $1 \leq i, j, k \leq m$ and $n > 1$, we have:

$$\begin{cases} p_{i,j,k}^{n+1} = \frac{1}{\gamma+1} (k^2 g_{i,j,k}^n + \alpha^2 (p_{i+1,j,k}^n + p_{i-1,j,k}^n + p_{i,j+1,k}^n + p_{i,j-1,k}^n + p_{i,j,k+1}^n + p_{i,j,k-1}^n) \\ \quad + (2 - 6\alpha^2 - \beta) p_{i,j,k}^n + (\gamma - 1) p_{i,j,k}^{n-1}) \\ p_{0,j,k}^n = p_{m+1,j,k}^n = p_{i,0,k}^n = p_{i,m+1,k}^n = p_{i,j,0}^n = p_{i,j,m+1}^n = 0, \\ p_{i,j,k}^0 = p_0(ih, jh, kh), p_{i,j,k}^1 = p_{i,j,k}^0 + kp_1(ih, jh, kh). \end{cases} \quad (36)$$

In this case, the eigenvalues of the matrix \mathcal{A}_{3D} are given by (see annex 7.4):

$$\lambda_{k,l,s}(\mathcal{A}_{3D}) = 4(\sin^2(\frac{k\pi h}{2}) + \sin^2(\frac{l\pi h}{2}) + \sin^2(\frac{s\pi h}{2})), 1 \leq k, l, s \leq m,$$

and the associated components of the eigenvectors are given by:

$$v_{i,j,q}^{k,l,s} = \sin(ik\pi h) \sin(jl\pi h) \sin(sq\pi h), 1 \leq i, j, q, k, l, s \leq m.$$

Consequently, expressed in the basis of eigenvectors, the homogeneous relation (35) is written as follows:

$$\bar{P}^{n+1} = \frac{1}{\gamma+1}((2-\beta)I - \alpha^2 \Lambda_{3D})\bar{P}^n + \frac{\gamma-1}{\gamma+1}\bar{P}^{n-1}.$$

Thus, by a similar way than for the two-dimensional case, we obtain the following relations:

$$\begin{pmatrix} \bar{P}^{n+1} \\ \bar{P}^n \end{pmatrix} = \begin{pmatrix} \frac{1.0}{\gamma+1}((2-\beta)\mathcal{I} - \alpha^2 \Lambda_{3D}) & \frac{\gamma-1}{1+\gamma}I \\ I & 0 \end{pmatrix} \begin{pmatrix} \bar{P}^n \\ \bar{P}^{n-1} \end{pmatrix},$$

and the eigenvalues λ of the amplification matrix thus satisfy the second degree equation:

$$\lambda^2 + \frac{(4\alpha^2(\sin^2(\frac{k\pi h}{2}) + \sin^2(\frac{l\pi h}{2}) + \sin^2(\frac{s\pi h}{2})) - 2 + \beta)}{1 + \gamma}\lambda + \frac{1 - \gamma}{1 + \gamma} = 0.$$

By using a similar way than in subsection 2.2, the stability condition is obtained when the sum of the roots is less than two. This leads to:

$$-2 \leq -\frac{(4\alpha^2(\sin^2(\frac{k\pi h}{2}) + \sin^2(\frac{l\pi h}{2}) + \sin^2(\frac{s\pi h}{2})) - 2 + \beta)}{1 + \gamma} \leq 2.$$

Thus similarly, we obtain the following result:

Proposition 29 *The explicit numerical scheme (36) being consistent and stable when*

$$\alpha \leq \frac{\sqrt[3]{3}}{3} \text{ and } \frac{\beta}{2} \leq \gamma,$$

is also convergent when $h \rightarrow 0$ and $k \rightarrow 0$.

4.3 Sixth explicit scheme.

Consider also the two-dimensional and the three-dimensional case when, as previously seen in subsection 2.3, the first derivative is discretized by a decenterd scheme. Similarly to the previous subsections and for the same reasons, we will limit ourselves to defining the numerical schemes and to studying the stability condition.

4.3.1 Two-dimensional damped wave equation.

In this case, for $1 \leq i, j \leq m$ and $n > 1$, the numerical scheme is defined as follows:

$$\begin{cases} \frac{p_{i,j}^{n+1} - 2p_{i,j}^n + p_{i,j}^{n-1}}{k^2} + d\left(\frac{p_{i,j}^{n+1} - p_{i,j}^n}{k}\right) - c^2\left(\frac{p_{i,j+1}^n + p_{i+1,j}^n - 4p_{i,j}^n + p_{i-1,j}^n + p_{i,j-1}^n}{h^2}\right) + \alpha p_{i,j}^n = g_{i,j}^n, \\ p_{0,j}^n = p_{m+1,j}^n = p_{i,0}^n = p_{i,m+1}^n = 0 \text{ and } p_{i,j}^0 = p_0(ih, jh), \frac{p_{i,j}^1 - p_{i,j}^0}{k} = p_1(ih, jh). \end{cases}$$

Let us multiply the previous relation by k^2 and consider the change of variable (11) with $\bar{\gamma} = d.k$. Finally, the numerical scheme is defined as follows, for $1 \leq i, j \leq m$ and $n > 1$:

$$\begin{cases} p_{i,j}^{n+1} = \frac{k^2}{\bar{\gamma}+1} g_{i,j}^n + \frac{\alpha^2}{\bar{\gamma}+1} (p_{i+1,j}^n + p_{i-1,j}^n + p_{i,j+1}^n + p_{i,j-1}^n) + \frac{\bar{\gamma}+2-4\alpha^2-\beta}{\bar{\gamma}+1} p_{i,j}^n - \frac{1}{\bar{\gamma}+1} p_{i,j}^{n-1}, \\ p_{0,j}^n = p_{m+1,j}^n = p_{i,0}^n = p_{i,m+1}^n = 0 \text{ and } p_{i,j}^0 = p_0(ih, jh), p_{i,j}^1 = p_{i,j}^0 + k p_1(ih, jh). \end{cases} \quad (37)$$

Once again, this previous relation can be written vectorially as follows:

$$P^{n+1} = \frac{k^2}{\bar{\gamma}+1} G^n + \frac{1}{\bar{\gamma}+1} ((\bar{\gamma}+2-\beta)\mathcal{I} - \alpha^2 \mathcal{A}_{2D}) \cdot P^n - \frac{1}{\bar{\gamma}+1} P^{n-1}.$$

According to the result mentioned in subsection 2.3 and by using the same way as previously, the truncation error is given by:

$$F \leq Q(k + h^2),$$

and once again, the explicit numerical scheme (37) is consistent and of order 1 in time and of order 2 in space.

For the study of the numerical stability of the explicit numerical scheme (37), we will use again the matricial method. Indeed, in the basis of eigenvectors of the matrix \mathcal{A}_{2D} , the following relation is obviously true in consideration of the homogeneous previous recurrence:

$$\begin{pmatrix} \bar{P}^{n+1} \\ \bar{P}^n \end{pmatrix} = \begin{pmatrix} \frac{1.0}{\bar{\gamma}+1} ((\bar{\gamma}+2-\beta)\mathcal{I} - \alpha^2 \Lambda_{2D}) - \frac{1.0}{1+\bar{\gamma}} \mathcal{I} \\ \mathcal{I} \\ 0 \end{pmatrix} \begin{pmatrix} \bar{P}^n \\ \bar{P}^{n-1} \end{pmatrix},$$

and the eigenvalues λ of the amplification matrix thus satisfy the second degree equation:

$$\lambda^2 - \frac{(\bar{\gamma}+2-4\alpha^2(\sin^2(\frac{k\pi h}{2}) + \sin^2(\frac{l\pi h}{2})) - \beta)}{1+\bar{\gamma}} \lambda + \frac{1}{1+\bar{\gamma}} = 0,$$

which involves that:

$$\alpha \leq \frac{\sqrt[3]{2}}{2} \text{ and } \frac{\beta}{3} \leq \bar{\gamma}.$$

Consequently, when the previous stability conditions are satisfied, the explicit numerical scheme is convergent when $h \rightarrow 0$ and $k \rightarrow 0$.

4.3.2 Three-dimensional damped wave equation.

In this case, for $1 \leq i, j, k \leq m$ and $n > 1$, the numerical scheme is given by:

$$\begin{cases} \frac{p_{i,j,k}^{n+1} - 2p_{i,j,k}^n + p_{i,j,k}^{n-1}}{k^2} + d \left(\frac{p_{i,j,k}^{n+1} - p_{i,j,k}^n}{k} \right) - c^2 \left(\frac{p_{i,j+1,k}^n + p_{i+1,j,k}^n - 6p_{i,j,k}^n + p_{i-1,j,k}^n + p_{i,j-1,k}^n}{h^2} \right) \\ - c^2 \left(\frac{p_{i,j,k+1}^n + p_{i,j,k-1}^n}{h^2} \right) + a p_{i,j,k}^n = g_{i,j,k}^n, \\ p_{0,j,k}^n = p_{m+1,j,k}^n = p_{i,0,k}^n = p_{i,m+1,k}^n = p_{i,j,0}^n = p_{i,j,m+1}^n = 0, \\ p_{i,j,k}^0 = p_0(ih, jh, kh), \frac{p_{i,j,k}^1 - p_{i,j,k}^0}{k} = p_1(ih, jh, kh), \end{cases}$$

which leads to:

$$\begin{cases} p_{i,j,k}^n = \frac{1}{\bar{\gamma}+1} (k^2 g_{i,j,k}^n + \alpha^2 (p_{i+1,j,k}^n + p_{i-1,j,k}^n + p_{i,j+1,k}^n + p_{i,j-1,k}^n + p_{i,j,k+1}^n + p_{i,j,k-1}^n) \\ \quad + (\bar{\gamma} + 2 - 6\alpha^2 - \beta) p_{i,j,k}^n - p_{i,j,k}^{n-1}), \\ p_{0,j,k}^n = p_{m+1,j,k}^n = p_{i,0,k}^n = p_{i,m+1,k}^n = p_{i,j,0}^n = p_{i,j,m+1}^n = 0, \\ p_{i,j,k}^0 = p_0(ih, jh, kh), \quad \frac{p_{i,j,k}^1 - p_{i,j,k}^0}{k} = p_1(ih, jh, kh). \end{cases}$$

This expression can be written vectorially as follows:

$$P^{n+1} = \frac{k^2}{\bar{\gamma}+1} G^n + \frac{1}{\bar{\gamma}+1} ((\bar{\gamma} + 2 - \beta) \mathcal{I} - \alpha^2 \mathcal{A}_{3D}) P^n - \frac{1}{\bar{\gamma}+1} P^{n-1}.$$

According to the result mentioned in subsection 2.3 and by using the same way, the truncation error is given by:

$$F \leq Q(k + h^2),$$

and once again, the previous explicit numerical scheme is consistent and of order 1 in time and of order 2 in space.

Let us consider now the study of the numerical stability by using the matricial method. In the basis of the set of eigenvectors of matrix \mathcal{A}_{3D} the homogeneous recurrence is written as follows:

$$\begin{pmatrix} \bar{P}^{n+1} \\ \bar{P}^n \end{pmatrix} = \begin{pmatrix} \frac{1-\beta}{\bar{\gamma}+1} ((\bar{\gamma} + 2 - \beta) \mathcal{I} - \alpha^2 \Lambda_{3D}) - \frac{1-\beta}{1+\bar{\gamma}} \mathcal{I} \\ \mathcal{I} \end{pmatrix} \begin{pmatrix} \bar{P}^n \\ \bar{P}^{n-1} \end{pmatrix},$$

and the eigenvalues λ of the amplification matrix thus satisfy the second degree equation:

$$\lambda^2 - \frac{(\bar{\gamma} + 2 - 4\alpha^2 (\sin^2(\frac{k\pi h}{2}) + \sin^2(\frac{l\pi h}{2}) + \sin^2(\frac{s\pi h}{2})) - \beta)}{1 + \bar{\gamma}} \lambda + \frac{1}{1 + \bar{\gamma}} = 0,$$

which involves:

$$\alpha \leq \frac{\sqrt[3]{3}}{3} \text{ and } \frac{\beta}{3} \leq \bar{\gamma}.$$

Note that, when the condition of the numerical stability are verified then, since the considered explicit schemes is consistent, then it is also convergent when $h \rightarrow 0$ and $k \rightarrow 0$.

4.4 Third implicit scheme.

In the case of implicit numerical scheme, we will consider now, in the two-dimensional and in the three-dimensionnal cases, the situation where the first dérivative is discretized by a centered scheme.

4.4.1 Two-dimensional damped wave equation.

In this case, for $1 \leq i, j \leq m$ and $n > 1$, the numerical scheme is defined by:

$$\left\{ \begin{array}{l} \frac{p_{i,j}^{n+1} - 2p_{i,j}^n + p_{i,j}^{n-1}}{k^2} + d \left(\frac{p_{i,j}^{n+1} - p_{i,j}^{n-1}}{2k} \right) - c^2 \left(\frac{p_{i,j+1}^{n+1} + p_{i+1,j}^{n+1} - 4p_{i,j}^{n+1} + p_{i-1,j}^{n+1} + p_{i,j-1}^{n+1}}{h^2} \right) + ap_{i,j}^{n+1} = g_{i,j}^{n+1}, \\ p_{0,j}^{n+1} = p_{m+1,j}^{n+1} = p_{i,0}^{n+1} = p_{i,m+1}^{n+1} = 0 \text{ and } p_{i,j}^0 = p_0(ih, jh), \frac{p_{i,j}^1 - p_{i,j}^0}{k} = p_1(ih, jh). \end{array} \right.$$

Let us multiply the previous relation by k^2 and let us consider also the change of variable (11), in such a way that the numerical scheme can be written as follows:

$$(4\alpha^2 + \beta + \gamma + 1)p_{i,j}^{n+1} - \alpha^2(p_{i+1,j}^{n+1} + p_{i-1,j}^{n+1} + p_{i,j+1}^{n+1} + p_{i,j-1}^{n+1}) = k^2 g_{i,j}^{n+1} + 2p_{i,j}^n + (\gamma - 1)p_{i,j}^{n-1}.$$

This relation can be written vectorially as follows:

$$((\gamma + 1 + \beta)I + \alpha^2 \mathcal{A}_{2D})P^{n+1} = k^2 G^{n+1} + 2P^n + (\gamma - 1)P^{n-1}, \quad (38)$$

where \mathcal{A}_{2D} is the discretization matrix of the Laplacian.

Concerning the truncation error, we obtain the same result than the one obtained in subsection 2.2, i.e.

$$F \leq Q(h^2 + k^2).$$

Thus, we have stated the following result:

Proposition 30 *The explicit numerical scheme (38) is consistent and of order 2 in time and space.*

For the study of the stability, we limit again ourselves to the matricial method. In the basis of the eigenvectors of matrix \mathcal{A}_{2D} the homogeneous relation (38) is written as follows:

$$\bar{P}^{n+1} = ((\gamma + 1 + \beta)I + \alpha^2 \Lambda_{2D})^{-1} (2\bar{P}^n + (\gamma - 1)\bar{P}^{n-1}),$$

and then, we obtain the following amplification matrix:

$$\mathcal{B} = \begin{pmatrix} \frac{2}{4\alpha^2(\sin^2(\frac{l\pi h}{2}) + \sin^2(\frac{k\pi h}{2})) + \gamma + \beta + 1} & \frac{(\gamma - 1)}{4\alpha^2(\sin^2(\frac{l\pi h}{2}) + \sin^2(\frac{k\pi h}{2})) + \gamma + \beta + 1} \\ 1 & 0 \end{pmatrix}.$$

Hence, the eigenvalues of \mathcal{B} satisfy the second degree equation:

$$\lambda^2 - \frac{2\lambda - (\gamma - 1)}{4\alpha^2(\sin^2(\frac{l\pi h}{2}) + \sin^2(\frac{k\pi h}{2})) + \gamma + \beta + 1} = 0. \quad (39)$$

Let us again remark that the sum of eigenvalues are equal to $\frac{2}{(4\alpha^2(\sin^2(\frac{l\pi h}{2}) + \sin^2(\frac{k\pi h}{2})) + \gamma + \beta + 1)}$ and is strictly positive. The product of the eigenvalues is equal to $\frac{(1 - \gamma)}{(4\alpha^2(\sin^2(\frac{l\pi h}{2}) + \sin^2(\frac{k\pi h}{2})) + \gamma + \beta + 1)}$. Let us consider two distinct cases:

- if $1 - \gamma > 0 \Leftrightarrow 1 > \gamma$ then the roots are of the same sign and moreover positive; the expression of the sum of the roots leads to the conclusion that, each eigenvalue has a modulus strictly less than one.

- if $1 - \gamma < 0 \Leftrightarrow 1 < \gamma$ then the roots are of opposite sign, but since their sum is positive, the larger of them in modulus is positive.

In both cases, the following inequalities:

$$-2 < \frac{2}{(4\alpha^2(\sin^2(\frac{l\pi h}{2}) + \sin^2(\frac{k\pi h}{2})) + \gamma + \beta + 1)} < 2,$$

are obviously verified and we can therefore conclude by giving the following Proposition:

Proposition 31 *The numerical implicit scheme (38) is unconditionally stable. Moreover, since the scheme is unconditionally stable and consistent, then it is convergent when $h \rightarrow 0$ and $k \rightarrow 0$.*

4.4.2 Three-dimensional damped wave equation.

In this case, for $1 \leq i, j, k \leq m$ and $n > 1$, the numerical scheme is defined by:

$$\begin{cases} (\gamma + 1 + \beta + 6\alpha^2)p_{i,j,k}^{n+1} - \alpha^2(p_{i+1,j,k}^{n+1} + p_{i-1,j,k}^{n+1} + p_{i,j+1,k}^{n+1} + p_{i,j-1,k}^{n+1} + p_{i,j,k+1}^{n+1} + p_{i,j,k-1}^{n+1}) \\ = k^2 g_{i,j,k}^{n+1} + 2p_{i,j,k}^n + (\gamma - 1)p_{i,j,k}^{n-1}, \\ p_{0,j,k}^{n+1} = p_{m+1,j,k}^{n+1} = p_{i,0,k}^{n+1} = p_{i,m+1,k}^{n+1} = p_{i,j,0}^{n+1} = p_{i,j,m+1}^{n+1} = 0, \\ p_{i,j,k}^0 = p_0(ih, jh, kh), p_{i,j,k}^1 = p_{i,j,k}^0 + kp_1(ih, jh, kh), \end{cases} \quad (40)$$

and it can be written vectorially as follows:

$$((\gamma + 1 + \beta)I + \alpha^2 \mathcal{A}_{3D})P^{n+1} = k^2 G^{n+1} + 2P^n + (\gamma - 1)P^{n-1}, \quad (41)$$

where \mathcal{A}_{3D} is the discretization matrix of the Laplacian.

The truncation error is equal to the one obtained in subsection 2.2, i.e.

$$F \leq Q(h^2 + k^2).$$

For the stability analysis, we are in the same situation than the one considered in the two-dimensional case and by using the same computations, in order to determine the amplification matrix, we find that the product of the eigenvalues of \mathcal{B} is equal to:

$$\frac{(1 - \gamma)}{(4\alpha^2(\sin^2(\frac{l\pi h}{2}) + \sin^2(\frac{k\pi h}{2}) + \sin^2(\frac{s\pi h}{2})) + \gamma + \beta + 1)},$$

while the sum of the eigenvalues is given by:

$$\frac{2}{(4\alpha^2(\sin^2(\frac{l\pi h}{2}) + \sin^2(\frac{k\pi h}{2}) + \sin^2(\frac{s\pi h}{2})) + \gamma + \beta + 1)},$$

and similarly, we conclude to the unconditional stability of the implicit numerical scheme (41).

Proposition 32 *The implicit numerical scheme (41) is consistent and unconditionally stable, of order 2 in space and in time and consequently convergent when $h \rightarrow 0$ and $k \rightarrow 0$.*

4.5 Sixth implicit scheme.

In this subsection, we consider the case where the first derivative in time is discretized by a decentered scheme. In this case, we will briefly formulate the numerical schemes and we indicate simply their properties, since the techniques for studying the stability of these implicit schemes are identical to those seen in the previous subsections.

4.5.1 Two-dimensional damped wave equation.

In this case, for $1 \leq i, j \leq m$ and $n > 1$, the numerical scheme is defined as follows:

$$\begin{cases} \frac{p_{i,j}^{n+1} - 2p_{i,j}^n + p_{i,j}^{n-1}}{k^2} + d\left(\frac{p_{i,j}^{n+1} - p_{i,j}^n}{k}\right) - c^2\left(\frac{p_{i,j+1}^{n+1} + p_{i+1,j}^{n+1} - 4p_{i,j}^{n+1} + p_{i-1,j}^{n+1} + p_{i,j-1}^{n+1}}{h^2}\right) + ap_{i,j}^{n+1} = g_{i,j}^{n+1}, \\ p_{0,j}^{n+1} = p_{m+1,j}^{n+1} = p_{i,0}^{n+1} = p_{i,m+1}^{n+1} = 0 \text{ and } p_{i,j}^0 = p_0(ih, jh), \frac{p_{i,j}^1 - p_{i,j}^0}{k} = p_1(ih, jh). \end{cases}$$

Let us multiply the previous relation by k^2 and consider the change of variable (11) with $\bar{\gamma} = dk$. Thus, the numerical scheme is defined as follows, for $1 \leq i, j \leq m$ and $n > 1$:

$$\begin{cases} (\bar{\gamma} + \beta + 4\alpha^2 + 1)p_{i,j}^{n+1} - \alpha^2(p_{i+1,j}^{n+1} + p_{i-1,j}^{n+1} + p_{i,j+1}^{n+1} + p_{i,j-1}^{n+1}) = k^2 g_{i,j}^{n+1} + (2 + \bar{\gamma})p_{i,j}^n - p_{i,j}^{n-1}, \\ p_{0,j}^{n+1} = p_{m+1,j}^{n+1} = p_{i,0}^{n+1} = p_{i,m+1}^{n+1} = 0 \text{ and } p_{i,j}^0 = p_0(ih, jh), \frac{p_{i,j}^1 - p_{i,j}^0}{k} = p_1(ih, jh). \end{cases} \quad (42)$$

Once again, this previous relation can be written vectorially as follows:

$$((\bar{\gamma} + \beta + 1)\mathcal{I} + \alpha^2 \mathcal{A}_{2D})P^{n+1} = k^2 G^n + (2 + \bar{\gamma})P^n - P^{n-1}.$$

According to the result mentioned in subsection 2.3 and by using the same way, the truncation error is given by:

$$F \leq Q(k + h^2),$$

and once again, the explicit numerical scheme (42) is consistent, of order 1 in time and of order 2 in space.

In order to study the stability of this numerical scheme, we consider the homogeneous equation written in the eigenvector basis of the matrix \mathcal{A}_{2D} . By using the matricial method, we obtain:

$$\bar{P}^{n+1} = ((\bar{\gamma} + \beta + 1)\mathcal{I} + \alpha^2 \Lambda_{2D})^{-1}((2 + \bar{\gamma})\bar{P}^n - \bar{P}^{n-1}).$$

This leads to the following amplification matrix:

$$\mathcal{B} = \begin{pmatrix} \frac{2 + \bar{\gamma}}{4\alpha^2(\sin^2(\frac{l\pi h}{2}) + \sin^2(\frac{k\pi h}{2})) + \bar{\gamma} + \beta + 1} & -\frac{1}{4\alpha^2(\sin^2(\frac{l\pi h}{2}) + \sin^2(\frac{k\pi h}{2})) + \bar{\gamma} + \beta + 1} \\ 1 & 0 \end{pmatrix},$$

whose eigenvalues are solutions of the following second-degree equation:

$$\lambda^2 - \frac{(2 + \bar{\gamma})\lambda - 1}{4\alpha^2(\sin^2(\frac{l\pi h}{2}) + \sin^2(\frac{k\pi h}{2})) + \bar{\gamma} + \beta + 1} = 0.$$

The product of the roots is equal to $\frac{1}{4\alpha^2(\sin^2(\frac{l\pi h}{2})+\sin^2(\frac{k\pi h}{2}))+\bar{\gamma}+\beta+1}$ and is positive so that, the eigenvalues have the same sign. In addition, the sum of the roots is equal to $\frac{2+\bar{\gamma}}{4\alpha^2(\sin^2(\frac{l\pi h}{2})+\sin^2(\frac{k\pi h}{2}))+\bar{\gamma}+\beta+1}$ and consequently, the eigenvalues are positive.

Due to the sign of $\bar{\gamma}$, β and α^2 , the conditions of stability can be easily verified and the implicit numerical scheme is unconditionally stable and convergent when $h \rightarrow 0$ and $k \rightarrow 0$ since it is consistent.

4.5.2 Three-dimensional damped wave equation.

Consider now the three-dimensional damped wave equation. By using the same notations, for $1 \leq i, j, k \leq m$ and $n > 1$, the considered discretization of this problem leads to:

$$\begin{cases} (\bar{\gamma} + \beta + 6\alpha^2 + 1)p_{i,j,k}^{n+1} - \alpha^2(p_{i+1,j,k}^{n+1} + p_{i-1,j,k}^{n+1} + p_{i,j+1,k}^{n+1} + p_{i,j-1,k}^{n+1} + p_{i,j,k+1}^{n+1} + p_{i,j,k-1}^{n+1}) \\ = k^2 g_{i,j}^{n+1} + (2 + \bar{\gamma})p_{i,j}^n - p_{i,j}^{n-1}, \\ p_{0,j,k}^{n+1} = p_{m+1,j,k}^{n+1} = p_{i,0,k}^{n+1} = p_{i,m+1,k}^{n+1} = p_{i,j,0}^{n+1} = p_{i,j,m+1}^{n+1} = 0, \\ p_{i,j}^0 = p_0(ih, jh), p_{i,j}^1 = p_{i,j}^0 + kp_1(ih, jh). \end{cases} \quad (43)$$

This leads to the following vectorial formulation:

$$((\bar{\gamma} + \beta + 1)\mathcal{I} + \alpha^2 \mathcal{A}_{3D})P^{n+1} = k^2 G^n + (2 + \bar{\gamma})P^n - P^{n-1}.$$

According to the result mentioned in subsection 2.3, by using the same way, the truncation error is given by:

$$F \leq Q(k + h^2),$$

and once again, the explicit numerical scheme (43) is consistent and of order 1 in time and of order 2 in space.

To study the stability of this numerical scheme, we consider the homogeneous equation written in the eigenvector basis of the matrix \mathcal{A}_{3D} . By using the matricial method, we obtain:

$$\bar{P}^{n+1} = ((\bar{\gamma} + \beta + 1)\mathcal{I} + \alpha^2 \Lambda_{3D})^{-1}((2 + \bar{\gamma})\bar{P}^n - \bar{P}^{n-1}),$$

which gives the following amplification matrix:

$$\mathcal{B} = \begin{pmatrix} \frac{2+\bar{\gamma}}{4\alpha^2(\sin^2(\frac{l\pi h}{2})+\sin^2(\frac{k\pi h}{2})+\sin^2(\frac{s\pi h}{2}))+\bar{\gamma}+\beta+1} & \frac{-1}{4\alpha^2(\sin^2(\frac{l\pi h}{2})+\sin^2(\frac{k\pi h}{2})+\sin^2(\frac{s\pi h}{2}))+\bar{\gamma}+\beta+1} \\ 1 & 0 \end{pmatrix},$$

whose eigenvalues are solutions of the following second degree equation:

$$\lambda^2 - \frac{(2 + \bar{\gamma})\lambda - 1}{4\alpha^2(\sin^2(\frac{l\pi h}{2}) + \sin^2(\frac{k\pi h}{2}) + \sin^2(\frac{s\pi h}{2})) + \bar{\gamma} + \beta + 1} = 0. \quad (44)$$

By proceeding analogously to the two-dimensional case, we conclude to the unconditional stability of the scheme. Thus, the implicit numerical scheme is unconditionally stable and convergent when $h \rightarrow 0$ and $k \rightarrow 0$ since it is consistent.

4.6 Numerical implicit scheme derived from the Gear scheme.

Lastly, let us consider now the case where the first derivative, with respect to the time, is discretized by the Gear scheme presented in subsection 2.6. In the two-dimensional and three-dimensional cases, we obtain the following truncation error:

$$F \leq Q(k^2 + h^2),$$

and the implicit scheme will be of order two in space and in time. Consequently, it is consistent.

4.6.1 Two-dimensional damped wave equation.

In the two-dimensional case, for $1 \leq i, j \leq m$ and $n > 1$, the scheme is defined as follows:

$$\begin{cases} \frac{p_{i,j}^{n+1} - 2p_{i,j}^n + p_{i,j}^{n-1}}{k^2} + d \left(\frac{3p_{i,j}^{n+1} - 4p_{i,j}^n + p_{i,j}^{n-1}}{2k} \right) - c^2 \left(\frac{p_{i,j+1}^{n+1} + p_{i+1,j}^{n+1} - 4p_{i,j}^{n+1} + p_{i-1,j}^{n+1} + p_{i,j-1}^{n+1}}{h^2} \right) \\ + ap_{i,j}^{n+1} = g_{i,j}^{n+1}, \\ p_{0,j}^n = p_{m+1,j}^n = p_{i,0}^n = p_{i,m+1}^n = 0, \\ p_{i,j}^0 = p_0(ih, jh), \frac{p_{i,j}^1 - p_{i,j}^0}{k} = p_1(ih, jh). \end{cases}$$

This relation can again be written vectorially as follows:

$$((3\gamma + \beta + 1)I + \alpha^2 \mathcal{A}_{2D})P^{n+1} = k^2 G^{n+1} + 2(1 + 2\gamma)P^n - (1 + \gamma)P^{n-1}. \quad (45)$$

Let us consider the associated homogeneous semidiscretized scheme in time exprimed in the basis of eigenvectors of \mathcal{A}_{2D} which finally leads to the the amplification matrix given by:

$$\mathcal{B} = \begin{pmatrix} \frac{2(1+2\gamma)}{4\alpha^2(\sin^2(\frac{l\pi h}{2}) + \sin^2(\frac{k\pi h}{2})) + 3\gamma + \beta + 1} & -\frac{1+\gamma}{4\alpha^2(\sin^2(\frac{l\pi h}{2}) + \sin^2(\frac{k\pi h}{2})) + 3\gamma + \beta + 1} \\ 1 & 0 \end{pmatrix}.$$

Consequently, the eigenvalues of the amplification matrix satisfy the following second degree equation:

$$\lambda^2 - \frac{2(1 + 2\gamma)\lambda + (1 + \gamma)}{4\alpha^2(\sin^2(\frac{l\pi h}{2}) + \sin^2(\frac{k\pi h}{2})) + 3\gamma + \beta + 1} = 0.$$

The product of the roots is positive, so that the eigenvalues have the same sign. In addition, the sum of the roots is positive, and the eigenvalues are positive. Thus, the implicit numerical scheme is stable when:

$$-2 \leq \frac{2(1 + 2\gamma)}{4\alpha^2(\sin^2(\frac{l\pi h}{2}) + \sin^2(\frac{k\pi h}{2})) + 3\gamma + \beta + 1} \leq 2.$$

We can easily see that these inequalities are always verified and we can conclude that the implicit scheme derived from the Gear scheme (45) is unconditionally stable. In addition, since this scheme is consistent, then it is also convergent when $h \rightarrow 0$ and $k \rightarrow 0$.

4.6.2 Threedimensional damped wave equation.

In the three-dimensional case, for $1 \leq i, j, k \leq m$ and $n > 1$, the scheme is defined as follows:

$$\begin{cases} \frac{p_{i,j,k}^{n+1} - 2p_{i,j,k}^n + p_{i,j,k}^{n-1}}{k^2} + d \left(\frac{3p_{i,j,k}^{n+1} - 4p_{i,j,k}^n + p_{i,j,k}^{n-1}}{2k} \right) - c^2 \left(\frac{p_{i,j+1,k}^{n+1} + p_{i+1,j,k}^{n+1} - 6p_{i,j,k}^{n+1} + p_{i-1,j,k}^{n+1} + p_{i,j-1,k}^{n+1}}{h^2} \right) \\ -c^2 \left(\frac{p_{i,j,k+1}^{n+1} + p_{i,j,k-1}^{n+1}}{h^2} \right) + ap_{i,j,k}^{n+1} = g_{i,j,k}^{n+1}, \\ p_{0,j,k}^n = p_{m+1,j,k}^n = p_{i,0,k}^n = p_{i,m+1,k}^n = 0, \\ p_{i,j,k}^0 = p_0(ih, jh, kh), \quad \frac{p_{i,j,k}^1 - p_{i,j,k}^0}{k} = p_1(ih, jh, kh). \end{cases} \quad (46)$$

This relation can be written vectorially as follows:

$$((3\gamma + \beta + 1)I + \alpha^2 \mathcal{A}_{3D})P^{n+1} = k^2 G^{n+1} + 2(1 + 2\gamma)P^n - (1 + \gamma)P^{n-1}.$$

and by repeating the same reasoning and computations as above, the stability condition is obtained by verifying the following conditions:

$$-2 \leq \frac{2(1 + 2\gamma)}{4\alpha^2(\sin^2(\frac{l\pi h}{2}) + \sin^2(\frac{k\pi h}{2}) + \sin^2(\frac{s\pi h}{2})) + 3\gamma + \beta + 1} \leq 2.$$

These inequalities are always verified and the considered scheme is unconditionally stable and convergent when $h \rightarrow 0$ and $k \rightarrow 0$ since it is consistent.

4.7 Overview of some algorithms to solve numerically high-dimensional sparse linear systems.

In order to solve the two-dimensional or the three-dimensional damped wave equation, we can choose between :

- on the one hand, an explicit time marching scheme that can be implemented as a recurrence. However, this type of method is limited by the constraints imposed by the stability conditions which have the effect of imposing relatively small time step choices. Nevertheless, it should be noted that, in order to reduce elapsed times, these methods are well suited to the use of vector processors and GPUs (Graphics Processing Units).
- on the other hand, an implicit time marching scheme, where at each time step we have to solve a very large and sparse linear algebraic system of the following type:

$$((\theta + 1)I + \alpha^2 \mathcal{A}).P = \tilde{G}, \quad (47)$$

where the positive real number θ results from the implicit numerical scheme presented in subsection 4.4 to 4.6 and where \mathcal{A} is the discretization matrix of the Laplacian. In the present subsection, we briefly review the main algorithms for solving this kind of algebraic system, focusing mainly on those that are well suited to the case where the matrix is large, sparse, symmetric, positive definite and strictly diagonal dominant (and even irreducible strictly diagonal dominant), which is typical when discretizing the Poisson equation by finite difference

method (or by finite volume method) on grids with very small discretization steps in space for good accuracy. We therefore limit ourselves to the target application concerning the numerical solution of the damped wave equation, and refer to [11, 12, 13, 14, 15, 16, 17, 18] for further informations.

In the case of two-dimensional or three-dimensional problems, among the methods for solving linear algebraic systems, we can already exclude direct methods such as Gauss method of elimination (and its variants), which requires $\frac{2}{3}M^3$ arithmetic operations for a full matrix of dimension M . Moreover, taking into account the symmetry of the matrix $((\theta + 1)I + \alpha^2 \mathcal{A})$ the number of arithmetic operations reduces to $\frac{1}{3}M^3$. Finally, taking into account the matrix band structure reduces the number of arithmetic operations to the order of Ml^2 , where l represents the half bandwidth. Furthermore, the discretization matrix of the Poisson equation is ill-conditioned and so, the propagation of rounding errors, due to the poor representation of real numbers in machine, can denature the result obtained numerically as M increases. This classical situation occurs when high approximation accuracy is required. Hence, despite the notable reduction in the number of arithmetic operations in the latter two situations, the effects of matrix ill-conditioning are only slightly attenuated, and this type of method is not usually used to solve this type of linear system [11, 12, 13, 14, 15].

Given the sparse structure of the matrix, we prefer to solve discretized systems using iterative methods, which essentially involve matrix-vector products and require fewer arithmetic operations. However, for our target application, we need to take into account the symmetry property of the matrix involved in the algebraic systems to be solved. In the following, we summarize the presentation of these methods in each case, and refer the reader to the literature [11, 12, 13, 14, 15] for further details.

4.7.1 Resolution by the relaxation method.

First, we consider point relaxation methods where, in order to solve the algebraic system (47) resulting from the discretization by finite difference schemes (or by finite volume schemes) of the Poisson equation, a fixed point problem of the following type is associated with the algebraic system to solve:

$$P = BP + Z.$$

This formulation results from the decomposition of $((\theta + 1)I + \alpha^2 \mathcal{A})$ into the form $((\theta + 1)I + \alpha^2 \mathcal{A}) = D - R - R^t$ where D is the diagonal of $((\theta + 1)I + \alpha^2 \mathcal{A})$ and $(-R)$ represents the lower strictly triangular part. We therefore distinguish :

- the point Jacobi method where $B_J \equiv J = D^{-1}(R + R^t)$ corresponding in fact to the method of successive approximations, the iteration being defined by :

$$P^{q+1} = B_J.P^q + Z, \text{ where } P^0 \text{ is given.}$$

- the point relaxation method where $B_\omega = (D - \omega R)^{-1}((1 - \omega)D + \omega R^t)$ corresponding to the following iteration:

$$P^{q+1} = B_\omega P^q + \omega(D - \omega R)^{-1}Z, \text{ where } P^0 \text{ is given and } 0 < \omega < 2,$$

with the particular case of the point Gauss-Seidel method when $\omega = 1$.

We consider also the block relaxation methods. Indeed, considering the block structure of the discretization matrix $((\theta + 1)I + \alpha^2\mathcal{A})$, we can also consider the block relaxation methods which are formally defined in a similar way to the point relaxation methods. Thus, the matrix D is replaced by a matrix \tilde{D} constructed from the diagonal blocks of the matrix $((\theta + 1)I + \alpha^2\mathcal{A})$ and where the matrix R is also replaced by the matrix \tilde{R} constructed also from the off-diagonal blocks of $((\theta + 1)I + \alpha^2\mathcal{A})$. For the block Jacobi method, we have:

$$\tilde{J} = \tilde{D}^{-1}(\tilde{R} + \tilde{R}^t),$$

and for the block relaxation method, we have:

$$\tilde{B}_\omega = (\tilde{D} - \omega\tilde{R})^{-1}((1 - \omega)\tilde{D} + \omega\tilde{R}^t),$$

where when $\omega = 1$, this defines the block Gauss-Seidel method.

Remark 17 *In the point Jacobi method, the components of the vector P^{q+1} are computed from the components of the previous iterate P^q , which must therefore be stored in memory until all the components of P^{q+1} have been computed. On the other hand, in the point Gauss-Seidel method and, more generally, in the point relaxation method, the i th component of P^{q+1} is calculated using the $(i-1)$ components of this vector that have already been updated and the $(M-i)$ components of P^q . In the latter case, there is a significant economy of memory allocation, since the old relaxed components are replaced by the new ones as the relaxations are updated. A priori, for the target problem under consideration, the Gauss-Seidel method converges twice as fast as the Jacobi one. A similar remark can be made in the case of block relaxation methods.*

In order to analyze the behavior of these relaxation methods, we classically recall, on the one hand, necessary and sufficient conditions for convergence linked to the spectral properties of the iteration matrices and, on the other hand, sufficient conditions for the convergence derived directly from the properties of the discretization matrix. We summarize the main convergence criteria below and refer to the literature for further informations.

- **Necessary and sufficient criteria for convergence of relaxation methods:** if we consider the problem (47) defined in a square (respectively cubic) domain Ω and discretized with a uniform discretization step in space h , the spectral radius of the point Jacobi matrix is given in the two-dimensional case by:

$$\rho(J) = \frac{4\alpha^2}{(4\alpha^2 + \theta + 1)} \cos(\pi h),$$

(and respectively in the three-dimensional case by:

$$\rho(J) = \frac{6\alpha^2}{(6\alpha^2 + \theta + 1)} \cos(\pi h).$$

Thus, in both cases $\rho(J) < 1$. Hence, we can check that $\rho(J)$ is less than 1, which ensures the convergence of the point Jacobi method whatever be the initial approximation P^0 . It should be noted that in the case of a domain Ω of any shape, the computation of the spectral radii of

the iteration matrices is not generally a straightforward problem, as it requires the implementation of the power method [11] and as it corresponds to a computational overhead. However, for a block tridiagonal matrix $((\theta + 1)I + \alpha^2\mathcal{A})$, we have other criteria which constitute sufficient conditions for the convergence. In this case, we have the following results (see [11]):

- let $((\theta + 1)I + \alpha^2\mathcal{A})$ be a tridiagonal block matrix whose diagonal blocks are invertible. Then the block Jacobi and block Gauss-Seidel methods converge or diverge simultaneously. In the case of convergence, the block Gauss-Seidel method converges asymptotically twice as fast as the block Jacobi method.

- Let $((\theta + 1)I + \alpha^2\mathcal{A})$ be a block tridiagonal matrix whose diagonal blocks are invertible. If μ is an eigenvalue of the block Jacobi matrix and if ξ verifies the following relation:

$$(\xi + \omega - 1)^2 = \xi \cdot \omega^2 \cdot \mu^2,$$

then ξ is an eigenvalue of the block relaxation matrix and, in particular for $\omega = 1$, $\xi = \mu^2$; this confirms the previous result concerning the convergence speed of the block Gauss-Seidel method compared with that of the block Jacobi method. Conversely, if ξ is a nonzero eigenvalue of B_ω and if μ verifies the previous relation, then μ is an eigenvalue of the block Jacobi matrix.

- Let $((\theta + 1)I + \alpha^2\mathcal{A})$ be a block tridiagonal matrix whose diagonal blocks are invertible. If all eigenvalues of the block Jacobi matrix are real, then the block Jacobi method and the block relaxation method (for $0 < \omega < 2$) converge or diverge simultaneously. In the case of convergence, there exists a value ω_{opt} defined by :

$$\omega_{opt} = \frac{2}{1 + \sqrt[2]{1 - \rho(\tilde{J})}},$$

for which $\rho(B_\omega)$ is minimal and equals to $\omega_{opt} - 1$.

- **Sufficient criteria for convergence of relaxation methods:** there are, however, sufficient convergence conditions linked to the properties of the discretization matrices. The great advantage of these criteria lies in the fact that it is not necessary to compute the eigenvalues of the iteration matrices, corresponding to a particularly difficult and delicate computation except in academic cases (see section 7). Note that the conditions, set out below, are obviously verified for the discretized problem arising from the damped wave equation.

- A first sufficient condition for convergence is satisfied when the matrix $((\theta + 1)I + \alpha^2\mathcal{A})$ is symmetric positive definite; in this case, we have the following results (see [11]):

- if and only if $((\theta + 1)I + \alpha^2\mathcal{A})$ is a symmetric positive definite matrix, the point and the block Gauss-Seidel methods converge. More generally, if $((\theta + 1)I + \alpha^2\mathcal{A})$, decomposed by points or by blocks, is symmetric with a positive diagonal D or a positive definite block diagonal \tilde{D} , then the relaxation methods by points or by blocks, with $0 < \omega < 2$, converge if and only if $((\theta + 1)I + \alpha^2\mathcal{A})$ is positive definite ;

- if $((\theta + 1)I + \alpha^2\mathcal{A})$ is symmetric and if $2D - ((\theta + 1)I + \alpha^2\mathcal{A})$ is symmetric positive definite then the point and block Jacobi methods converge if and only if $((\theta + 1)I + \alpha^2\mathcal{A})$ is positive definite.

- A second sufficient condition for convergence is satisfied when the matrix \mathcal{A} has strict diagonal dominance properties, without necessarily being symmetric. We have the following results (see [11]):

- let $((\theta + 1)I + \alpha^2\mathcal{A})$ be a matrix strictly dominant diagonal, then the point Jacobi method converges;

- let $((\theta + 1)I + \alpha^2\mathcal{A})$ be a strongly diagonal dominant matrix, then the point relaxation method, with $0 < \omega < 2$, converges.

Remark 18 *Note that the two sufficient convergence conditions apply directly to the solution of the discretized linear system derived from the damped wave equation. We also note that the strict diagonal dominance condition is very strong and can be replaced by a weaker condition consisting of an irreducible diagonal dominance property which is not necessary in our application, even if the matrix is irreducible.*

4.7.2 Resolution by the alternating directions method.

Another iterative relaxation method is the alternating directions method. We always consider the discretization matrix $((\theta + 1)I + \alpha^2\mathcal{A})$ in the case where the domain Ω is of two-dimensional (or three-dimensional) shape. In the first case, i.e. for the two-dimensional problem, we can decompose the discretization matrix into the form $((\theta + 1)I + \alpha^2\mathcal{A}) = \mathcal{A}_1 + \mathcal{A}_2$ where the matrices \mathcal{A}_1 and \mathcal{A}_2 refer to the discretization schemes of the second derivatives with respect to x and y respectively, for the two-dimensional problem. Depending on the numbering of the mesh points, for example when the mesh points are numbered in a direction parallel to the Ox axis, then \mathcal{A}_1 is block tridiagonal matrix whereas \mathcal{A}_2 is not. However, let \bar{Q} be the permutation matrix corresponding to the renumbering of the points in the domain parallel to the Oy axis, rather than parallel to the Ox axis as previously; then, we have $\mathcal{A}_2 = \bar{Q}^t \mathcal{A}_2 \bar{Q}$ and, under these conditions, \mathcal{A}_2 is a block tridiagonal matrix. Solving systems of the type $\mathcal{A}_1 P = Z_1$ and $\mathcal{A}_2 P = Z_2$ can therefore be decomposed into solving tridiagonal algebraic systems of small dimension using the TDMA algorithm, with the advantage that this algorithm leads to very fast solving and requires no additional memory storage. For the model problem (47), let r be a real number; so, let us write:

$$(\mathcal{A}_1 + rI)P = \tilde{G} + (rI - \mathcal{A}_2)P,$$

and

$$(\mathcal{A}_2 + rI)P = \tilde{G} + (rI - \mathcal{A}_1)P.$$

Each iteration of the alternating directions method consists of the following two predictor-corrector steps:

$$(\mathcal{A}_1 + rI)P^{q+\frac{1}{2}} = \tilde{G} + (rI - \mathcal{A}_2)P^q,$$

and

$$(\mathcal{A}_2 + rI)P^{q+1} = \tilde{G} + (rI - \mathcal{A}_1)P^{q+\frac{1}{2}},$$

where the number r is to be determined as well as possible. Note that the error between the exact solution of the problem P and P^q , i.e. e^q , on the one hand, and between P and P^{q+1} , i.e. e^{q+1} , on the other hand, can be written as follows:

$$e^{q+1} = S(r).e^q,$$

where

$$S(r) = (\mathcal{A}_2 + rI)^{-1} \cdot (rI - \mathcal{A}_1) \cdot (\mathcal{A}_1 + rI)^{-1} \cdot (rI - \mathcal{A}_2).$$

Since the matrices \mathcal{A}_1 and \mathcal{A}_2 commute their eigenvalues λ_{min} and λ_{max} are identical and it has been shown (see [11]) that the optimal value of r , denoted r_{opt} , which minimizes the spectral radius of $S(r)$ is given by $r_{opt} = \sqrt[2]{\lambda_{min} \cdot \lambda_{max}}$.

Remark 19 *Note that the alternating directions method is not very interesting to implement, since it requires two to three times as many operations per iteration than the optimal parameter point-relaxation method. However, we shall see in subsection 4.7.5 that this method can be easily parallelized.*

4.7.3 Resolution by the multigrid method.

Before presenting the multigrid method, let us present the 2-grid method. This method combines two poorly performing methods to create a highly efficient one. In the 2-grid method, we consider a fine grid of the domain Ω with step size h and a coarse grid with step size $H = 2h$. A cycle of the 2-grid method consists of the following two phases:

- **a smoothing phase**, corresponding to two or three iterations of a relaxation or other iterative method, to reduce the high frequencies of the error when decomposing it into the eigenvectors basis of the matrix;
- **a coarse-grid correction phase** that effectively handles low frequency errors, since a low frequency mode on the fine grid will eventually be transformed into an oscillating mode (corresponding to high frequencies) on the coarse grid, and will therefore be smoothed by an iterative smoothing method, as indicated in the previous point.

We can verify that the correction to be added to the result on the h -step grid is the solution of a problem of the same type as the initial problem, in which the second member is replaced by the residue at the points on the fine grid. The computation of this correction is, a priori, as costly as computing the solution to the initial problem. However, since the error is smoothed, we can try to obtain an approximation of this correction on the H -step grid which will require much less computations, since the H -step grid has about four times fewer points in the case of the two-dimensional problem (eight times fewer points in the case of the three-dimensional problem) than the h -step grid. We will indeed be able to obtain a good approximation of the correction on the coarse grid if it varies slowly which is effectively the case thanks to the smoothing iterations. The second phase of each cycle of the 2-grid method then consists of :

- 1) Define the restriction of the residue on the coarse grid by using a restriction operator which can simply be the injection defined at all points on the coarse grid, or a weighted-average restriction,
- 2) Solve the linear system to obtain the correction on the coarse grid,
- 3) Extend the correction obtained on the coarse grid to the fine grid, for example by bilinear interpolation. We then repeat a new cycle of the 2-grid method.

The main question is whether the extension of the correction obtained on the coarse grid is a good approximation of the correction on the fine grid, and if so, what benefits can be expected from using the 2-grid method. Note that the 2-grid method can be written as a linear fixed-point iteration of the type $P^{q+1} = BP^q + Z$, for $q = 0, 1, 2, \dots$. For smoothing corresponding to the Gauss-Seidel method, we can show by a Fourier analysis that the 2-grid method yields an error reduction factor equal to $\sqrt[3]{5} = 0.447$ which corresponds to an excellent error reduction rate.

The multigrid method corresponds to the recursive application of a 2-grid method. Indeed, when solving the system of equations on the coarse grid to obtain the correction (point number 2 just above), we can once again use a 2-grid method, defining a problem on an still coarser grid of steps $2H = 4h$, and so on, to finally solve a system of equations on the coarsest grid containing just few points, or even a single point since the discretization steps are chosen as inverse powers of 2. This requires the definition of inter-grid passage strategies, which we will not develop here as the reader will find many details in the literature (see [17, 18]). It should also be noted that this multigrid method is well suited to use on rectangular domains in the two-dimensional case or cubic domains in the three-dimensional case. However, there are variants of the multigrid method well suited to the numerical solution of partial differential equations on domains of any shape.

4.7.4 Resolution by the conjugate gradient method.

In such method, we have to minimize a positive definite quadratic form. By derivation, we verify that, under the assumptions of symmetry of matrix \mathcal{A} , the minimum of this quadratic form is the solution of the linear system (47) resulting from the discretization of the damped wave equation to be numerically solved. The basic algorithms consist of the deepest descent method, the constant optimal parameter gradient method (or Richardson method). Let $\kappa = \frac{\lambda_{max}}{\lambda_{min}}$ be the condition number of the matrix $((\theta+1)I + \alpha^2\mathcal{A})$, where λ_{min} and λ_{max} represent respectively the largest and smallest eigenvalues of this matrix.

Note that for problem (47), we can give an estimate of the condition number κ when the discretisation step h tends towards zero. Indeed in the two-dimensional case, κ is given by:

$$\kappa = \frac{\theta + 1 + 8\alpha^2 \sin^2(\frac{m\pi h}{2})}{\theta + 1 + 8\alpha^2 \sin^2(\frac{\pi h}{2})},$$

while in the three-dimensional case κ is given by

$$\kappa = \frac{\theta + 1 + 12\alpha^2 \sin^2(\frac{m\pi h}{2})}{\theta + 1 + 12\alpha^2 \sin^2(\frac{\pi h}{2})}.$$

However, the number α depends on the parameter h . Hence, in the two-dimensional case and when h tends to zero, we can replace in the denominator $\sin^2(\frac{\pi h}{2})$ by $(\frac{\pi h}{2})^2$ and since $\alpha = \frac{ck}{h}$ the denominator is reduced to $\theta + 1 + 2(c\pi k)^2$. Thus, we obtain:

$$\lim_{h \rightarrow 0} (\kappa) \approx \left(\frac{\theta + 1 + 8\alpha^2 \sin^2(\frac{m\pi h}{2})}{\theta + 1 + 2(c\pi k)^2} \right) = \left(\frac{\theta + 1 + 8(\frac{ck}{h})^2 \sin^2(\frac{m\pi h}{2})}{\theta + 1 + 2(c\pi k)^2} \right),$$

and since m becomes large when h tends towards zero, we obtain the following overestimation of κ :

$$\kappa < \left(\frac{\theta + 1 + 8(\frac{ck}{h})^2}{\theta + 1 + 2(c\pi k)^2} \right) \rightarrow \infty \text{ when } m \rightarrow \infty.$$

We then verify that κ is greater than one and that the constraint $\kappa \geq 1$ is verified when m verifies $m > 1.570795L - 1$ since in our case m must be very large and obviously the constraint $\kappa > 1$ is well verified.

In a similar way, in the three-dimensional case, with similar estimates, we obtain:

$$\lim_{h \rightarrow 0}(\kappa) \approx \left(\frac{\theta + 1 + 12\left(\frac{ck}{h}\right)^2 \sin^2\left(\frac{m\pi h}{2}\right)}{\theta + 1 + 3(c\pi k)^2} \right) < \left(\frac{\theta + 1 + 12\left(\frac{ck}{h}\right)^2}{\theta + 1 + 3(c\pi k)^2} \right),$$

and, as the previous condition $\kappa \geq 1$ involves, once again, that m must verify $m > 1.570795L - 1$.

For these two previous methods, i.e. the deepest descent method and the Richardson method, the error reduction factor at each iteration is then equal to $\frac{\kappa-1}{\kappa+1}$. However, for large dimension M , this number κ increases considerably, and since $\ln(1 \pm \tau) \approx \pm\tau$, the number of iterations q for convergence is given by (see [11]):

$$q \approx \frac{\kappa}{4} \cdot \ln\left(\frac{1}{\epsilon}\right),$$

where ϵ represents the threshold of the iteration stop test. Hence, the number q of iterations to reach convergence is proportional to κ and it becomes as large as the condition number.

In order to reduce the computation time, it is preferable to use the conjugate gradient method, in which the direction of descent is corrected at each iteration. In exact arithmetic this method converges in at most M iterations; the error reduction factor of the conjugate gradient method is equal to $\frac{\sqrt[2]{\kappa}-1}{\sqrt[2]{\kappa}+1}$. Thus, after q iterations of this method, if the threshold of the iteration stop test is still set to ϵ , the number q is given by (see [11]):

$$q \approx \frac{\sqrt[2]{\kappa}}{4} \cdot \ln\left(\frac{4}{\epsilon}\right).$$

Thus, in this case, the number q of iterations is proportional to $\sqrt[2]{\kappa}$ and the use of the conjugate gradient method is preferable to employ than the descent method or Richardson one.

The convergence of the conjugate gradient algorithm can be further accelerated by considering the preconditioned conjugate gradient method. This is achieved by preconditioning the matrix $((\theta + 1)I + \alpha^2 \mathcal{A})$ which consists in replacing the resolution of the initial linear algebraic system (47) by:

$$C^{-1}((\theta + 1)I + \alpha^2 \mathcal{A})P = C^{-1}\tilde{G}.$$

This system can still be written as:

$$C^{\frac{1}{2}}(C^{-1}((\theta + 1)I + \alpha^2 \mathcal{A}))C^{-\frac{1}{2}}C^{\frac{1}{2}}P = C^{-\frac{1}{2}}\tilde{G},$$

where the preconditioning matrix C^{-1} is chosen symmetric positive-definite, so that, the equivalent system to be solved is given by:

$$\tilde{\mathcal{A}}\tilde{P} = \tilde{G},$$

where the matrix $\tilde{\mathcal{A}} = C^{-\frac{1}{2}}((\theta + 1)I + \alpha^2\mathcal{A})C^{-\frac{1}{2}}$ is symmetric positive definite and similar to the matrix $C^{-1}((\theta + 1)I + \alpha^2\mathcal{A})$. The matrix C is chosen so that:

$$\kappa(\tilde{\mathcal{A}}) \ll \kappa((\theta + 1)I + \alpha^2\mathcal{A}).$$

However, the determination of the matrix C is a non-trivial problem.

The ideal value of $\tilde{\mathcal{A}}$ would result in $\kappa(\tilde{\mathcal{A}}) \equiv 1$ and, in practice, $\tilde{\mathcal{A}}$ is an approximation of the identity matrix, the matrix C^{-1} being an approximation of $((\theta + 1)I + \alpha^2\mathcal{A})^{-1}$. In practice, we have to find C^{-1} as close as possible to $((\theta + 1)I + \alpha^2\mathcal{A})^{-1}$ without making a large lot of calculations to calculate C^{-1} , in practice too costly. If we rewrite the preconditioned conjugate gradient algorithm for solving the linear system (47), we can express the computational steps directly from the initial given matrix $((\theta + 1)I + \alpha^2\mathcal{A})$ and vectors P and \tilde{G} . We refer to [11, 12, 13, 14, 15] for the expression of this algorithm expressed directly from the data $((\theta + 1)I + \alpha^2\mathcal{A})$, P and \tilde{G} . Simply note that the expression of the preconditioned conjugate gradient algorithm is almost identical to that of the expression of the classical conjugate gradient method. The only difference is the appearance of linear systems of the type $CX = Y$ where X and Y are vectors of the same dimension than P and \tilde{G} . It is therefore absolutely essential that this resolution is easy and inexpensive which will be the case if the matrix C is a diagonal matrix or the product of triangular matrices. Classic choices of preconditioning matrices include :

- **The diagonal preconditioning** where $C = \text{diag}(((\theta + 1)I + \alpha^2\mathcal{A})) = D$ which in practice is not very efficient.
- **The Evans S.S.O.R. preconditioning** where $((\theta + 1)I + \alpha^2\mathcal{A}) = D - R - R^t$ with

$$C = \frac{1}{\omega(2 - \omega)}(D - \omega R)D^{-1}(D - \omega R^t),$$

where R is always the strictly triangular lower part of $((\theta + 1)I + \alpha^2\mathcal{A})$ and ω is a strictly positive real parameter located in $]0, 2[$. It can be seen that, in this case, C is obtained directly from $((\theta + 1)I + \alpha^2\mathcal{A})$. No additional storage or calculation are required. Solving the system $CX = Y$ requires solving a lower triangular system, then a diagonal system and an upper triangular system which leads to a sequence of very straightforward calculations. We can also determine the optimal value of ω given by (see [19] - [20]):

$$\omega_{opt} = \frac{2}{1 + \sqrt{1 - \rho(J)}},$$

where J is the point Jacobi matrix, ω_{opt} being the value of ω that minimizes the value of the condition number.

- **The preconditioning based on incomplete Cholesky factorization** : note that the matrix \mathcal{A} is symmetric and positive-definite, $((\theta + 1)I + \alpha^2\mathcal{A})$ admits a Cholesky decomposition of type $\tilde{S}\tilde{S}^t$ where \tilde{S} is a lower triangular matrix. To perform this type of preconditioning, we use a variant of the Cholesky method. In the case of a banded sparse matrix, the matrix \tilde{S} does not conserve the sparse character of the initial matrix, because during the factorization there is a "fill-in" phenomenon inside the band.

However, after the Cholesky transformation, as we move away from the main and most exterior diagonals, the moduli of the coefficients in this area become smaller and smaller, with smallest terms in the middle of the band. Such smallest terms are negligible compared to the diagonal and exterior diagonal terms. This suggests that some of these small terms should be neglected, and that the structure of the matrix S should be practically similar as the initial matrix $((\theta + 1)I + \alpha^2\mathcal{A})$. We therefore define, a priori, set of fixed indices σ and look for two matrices C (preconditioning matrix) and R such that:

$$((\theta + 1)I + \alpha^2\mathcal{A}) = C - R = SS^t,$$

where S is a lower triangular matrix, such that $S_{i,j} = 0$ if $(i, j) \in \sigma$ with $R_{i,j} \neq 0$ and $S_{i,j} \neq 0$ if $i \neq j$ or $R_{i,j} = 0$. Therefore, σ defines the set of a priori zero elements of S and non-zero elements of R . We can determine the matrix S , writing that the product SS^t corresponds to the exact decomposition of the matrix $((\theta + 1)I + \alpha^2\mathcal{A}) + R$.

We know how to characterize the classes of matrices for which the previous Cholesky decomposition, known as the Cholesky incomplete decomposition, is stable whatever σ . This is the case, for example, if $((\theta + 1)I + \alpha^2\mathcal{A})$ is strictly dominant diagonal which is the case for solving the damped wave equation (or also is an irreducible dominant diagonal matrix), and for verifying the following two properties: the off-diagonal entries of $((\theta + 1)I + \alpha^2\mathcal{A})$ are negative or zero and its inverse is nonnegative. All these properties are verified for matrices resulting from the discretization of the Poisson problem with Dirichlet or mixed boundary conditions and also, obviously, for the matrix resulting from the discretization of the damped wave equation. Then, such matrices are called classically M-matrices (see for example [16]). By varying σ , we obtain an infinite number of incomplete decompositions. The disadvantage of Cholesky incomplete factorization is that, in general, it cannot be shown to improve the condition number of the matrix $((\theta + 1)I + \alpha^2\mathcal{A})$. Nevertheless, numerical tests show that this is indeed the case, without, in general, any clear mathematical reasons.

4.7.5 Parallel and subdomain methods.

As already mentioned, to obtain good accuracy, very small spatial discretization steps must be chosen, leading to very large discretization matrices. Given the current evolution of computer architectures and, in particular the emergence of multiprocessor machines, the computations can be parallelized. Then, we have:

- when using conjugate gradient methods, matrix-vector products can be parallelized, as well as the scalar product of two vectors and the linear combination of two vectors;
- when using Jacobi point or block method, computations can also be parallelized;
- note also that, when using the alternating directions method, both the prediction and correction phases involve solving independent tridiagonal linear algebraic systems, making it easy to parallelize each of these phases;
- it should also be noted that some phases of the multigrid method can also be parallelized;
- finally, when using the Gauss-Seidel point or block method, calculations can also be parallelized with synchronous or asynchronous communications data exchanges between the parallel calculation processes, leading to the definition of the iterative subdomain method with or without overlaps between them, in the first case defining subdomain methods with overlaps, such as Schwarz alternating method. In this case, we can obtain an estimate of the

contraction constant $\nu(\mathcal{J})$ of the iterations derived from a contraction matrix \mathcal{J} (corresponding in fact in an appropriate mathematical context to a matricial Lipchitz constant (for more details see [21, 22, 23, 24]). In this framework, we have classically:

$$\nu(\mathcal{J}) \leq \|\mathcal{J}\|_{\infty},$$

where $\nu(\mathcal{J})$ is the spectral radius of \mathcal{J} and $\|\mathcal{J}\|_{\infty}$ is the matrix norm induced by the uniform norm which verifies:

$$\|\mathcal{J}\|_{\infty} \leq \max_{1 \leq i \leq M} \left(\sum_{j=1}^M |\mathcal{J}_{i,j}| \right).$$

In our case, in the two-dimensional case, the diagonal entries of \mathcal{J} are zero and its nonzero entries are given by $\frac{\alpha^2}{(4\alpha^2 + \theta + 1)}$. Then, we have:

$$\nu(\mathcal{J}) = \nu_{2D} < \frac{4\alpha^2}{(4\alpha^2 + \theta + 1)}.$$

In the similar way, in the three-dimensional case, we obtain:

$$\nu(\mathcal{J}) = \nu_{3D} < \frac{6\alpha^2}{(6\alpha^2 + \theta + 1)}.$$

In both cases, these estimates prove the convergence of these subdomains methods and $\nu(\mathcal{J})$ gives an estimate of the speed of convergence. For more details on this point, the reader is referred to [21, 22, 23, 24].

The previous methods being iterative in nature must be stopped when the iterated values are stabilized. Thus, they are completed by a test to stop the iterations. In the case of synchronous parallel algorithms, this iteration stopping test is identical to the one implemented in the sequential case. For asynchronous parallel algorithms, the iteration stopping test is much more delicate to implement. Thus, in this latter case, we will distinguish the computer science approach and the numerical analysis approach.

By concerning the computer science approach several methods are available by using a snapshot technic or also a token circulation (for more details see [21, 22, 23, 24]). Concerning the numerical analysis approach it can be proved that the successive iterates are located in nested sets centered in the solution of the problem (see [25] and [23]) which allows to implement efficient numerical stopping tests as soon as one can give an estimate of the diameters of each nested set and stop the iterative process when a diameter is smaller than a given tolerance. Such stopping tests have been implemented successfully on grids constituted by heterogeneous and distant machines. Note that asynchronous algorithms were very efficient when there was a lot of synchronizations between the processors (see [24]).

Remark 20 *Let us mention also a study on solving the telegrapher equation concerning the implementation of the optimized waveform relaxation method corresponding to a domain decomposition method for solving time dependent problems (see [26] for more details).*

4.8 Some non-linear situations of the damped wave equation.

In the present subsection, we consider two cases where the damped wave equation is pseudo-linear. The first case corresponds to the case where the damped wave equation is perturbed by a diagonal monotone operator while the second case concerns the situation where the solution of the damped wave equation is subject to inequality constraints. In this last situation, we have to solve a multivalued system.

Remark 21 We refer to [27] for a distinct study of a three level finite difference scheme of $O(k^4 + k^2 \cdot h^2 + h^4)$ using 9 grid points for the numerical solution of the one-dimensional non-linear wave equation defined with variable coefficients. Nevertheless, note that in [27] the non-linearity appears in the right hand side, while in the presented study, the non-linearity is not the same. Indeed in the present subsection we consider also, for example a multivalued perturbation of the linear operator or also situations where the perturbed operator is not differentiable.

4.8.1 Perturbation of the damped wave equation by a diagonal monotone operator.

We consider the first type of pseudo-linear singlevalued algebraic systems defined as follows:

$$((\theta + 1)I + \alpha^2 \mathcal{A})P + \Phi(P) = \tilde{G}. \quad (48)$$

It corresponds to a linear system perturbed by a diagonal increasing operator (or more generally monotone operator). By considering this property of monotony of the operators $P \rightarrow \Phi(P)$, the convergence criteria of the synchronous and asynchronous parallel algorithms, previously stated in the linear case, are still valid for the solution of the non-linear problem (48) (see [21]).

Concerning the non-linear part, we consider different types of non-linear problems where the non-linearities arise on the domain Ω . For example we can consider the following non-linear functions: $\varphi(p) = e^{\alpha p}$, with $\alpha > 0$ or $\varphi(p) = \log(\beta + \delta p)$, with $\delta > 0$, and a suitable sign for β , where in both cases $p \rightarrow \varphi(p)$ is a continuous non-decreasing function. Then, we have to solve an algebraic system like (48) where Φ is a diagonal operator derived from the discretization of φ ; according to the properties of φ then Φ is a monotone diagonal increasing (or monotone) mapping. Consequently, we are in the framework presented in [21] and the convergence of parallel synchronous or asynchronous relaxation methods is verified with the same speed of convergence than in the linear case since the properties of the mapping $\varphi(p)$, i.e. the property of monotony, do not change the convergence rate. Note that in this case, when the operator $p \rightarrow \varphi(p)$ is differentiable, Newton method can be used.

We can also consider another type of non-linear problem where the non-linearity is shown in Figure 1. This figure displays some examples of graphs for function φ . In particular, the two first graphs model saturation phenomena and the third graph models a multivalued function corresponding to the condition occurring when p is subject to constraints inequality. We consider first the resolution of the non-linear damped wave problem by limiting to a non-linearity corresponding to the two first graphs of Figure 1. Indeed the non-linearity presented in the last graph of this figure, corresponding to the resolution of a problem with inequality constraints will be studied in the next subsection 4.8.2. It should be noted that the applications presented in the first two graphs of Figure 1, are not differentiable everywhere, especially at

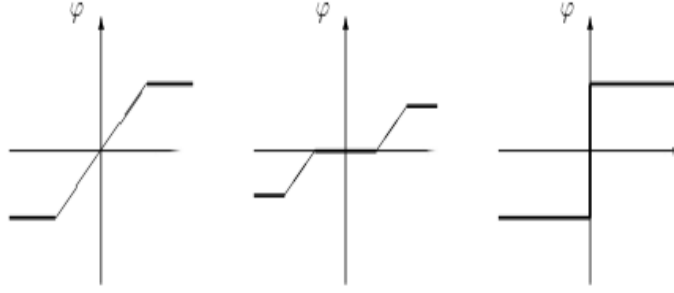


Fig. 1: Different graphs for φ .

the junction of the parts where φ is constant with its increasing parts. Therefore, the use of Newton method is not possible in this case. On the other hand, we can consider the use of a successive approximation method particularly well adapted to the solution of the considered type of problem in this case when the fixed point associated to the problem (48) is contracting in such a way that the parallel algorithm converges.

Concerning the convergence of the considered method, since on one hand, the diagonal entries $a_{i,i}$ of $((\theta + 1)I + \alpha^2 \mathcal{A})$ are strictly positive and the off-diagonal entries $a_{i,j}$ are non-positive and furthermore, since $((\theta + 1)I + \alpha^2 \mathcal{A})$ is strictly (or also irreducibly diagonally dominant) thus, $((\theta + 1)I + \alpha^2 \mathcal{A})$ is an M -matrix. On the other hand, since $\Phi(u)$ is a monotone increasing mapping then for the resolution of problem (48) the parallel synchronous or asynchronous relaxation algorithms converge to the unique solution of the problem and the estimate of the previous contraction constant $\nu(\mathcal{J})$ is at least still valid for the problem (48). Note also that, in the considered framework, the value of the contraction constant $\nu(\mathcal{J})$ are still valid when subdomain methods without overlapping are implemented (see [21]). Similarly, if we solve the non-linear simultaneous equations via the Schwarz alternating method, then the augmentation process of the Schwarz alternating method transforms the M -matrix \mathcal{A} into an M -matrix $\tilde{\mathcal{A}}$ and the monotone increasing mapping Φ into the monotone increasing mapping $\tilde{\Phi}$ (see [28] and [21]). Consequently, we are in the same previous framework considered in the case of subdomain methods without overlapping and the computational process converges too.

4.8.2 Situation where the solution of the damped wave equation is subject to inequality constraints.

Another situation of pseudolinear problem corresponds to the case where the solution P is subject to the following constraints of inequalities type:

$$P_{min} \leq P \text{ or } P_{min} \leq P \leq P_{max} \text{ or } P \leq P_{max}, \quad (49)$$

in this last case, the solution P satisfies the following large pseudo-linear multivalued algebraic systems:

$$((\theta + 1)I + \alpha^2 \mathcal{A})P + \partial\Psi(P) - G \ni 0, \quad (50)$$

where Ψ is the characteristic function defining the convex set of constraints and $\partial\Psi(P)$ is the subdifferential of Ψ corresponding, in fact, to a weak derivative of Ψ . Classically the mapping $P \rightarrow \partial\Psi(P)$ is still a diagonal monotone operator. Then, we are in an analogous framework considered just above in the singlevalued situation where a linear algebraic system is perturbed by a diagonal monotone operator. Thus, the parallel synchronous or asynchronous relaxation methods applied to the resolution of problem (49) converge to the unique solution of the problem (see [21]).

5 Numerical simulation for 1D damped wave equation.

In order to validate our study, in particular the development of various explicit and implicit schemes presented above, we consider in this section numerical tests in the one-dimensional case only. To test these schemes, we place ourselves in situations where we know a priori the solution to the problem. We consider two distinct situations studied respectively in sections 2 and 3. In the case of implicit schemes, we have to solve a linear system composed with a tridiagonal matrix, and given the sparse nature of this matrix, we will use the TDMA method.

5.1 Damped wave equation equipped with Dirichlet conditions.

In the present subsection, we consider the damped wave equation equipped with Dirichlet boundary conditions. In the considered test $c = 1$ and $a = d = 2$. The exact solution is given by $p_{exact}(x, t) = \sin(\pi.x) \cdot \cos(\pi.t)$ and the second member is defined by $g(x, t) = \sin(\pi.x) \cdot (\cos(\pi.t) \cdot (a + \pi^2 \cdot (c^2 - 1)))$. Note that $p_{exact}(0, t) = p_{exact}(L, t) = 0$ with $L = 1$, and consequently $p_{exact}(x, t)$ satisfies homogeneous Dirichlet boundary conditions. In addition, the initial condition is given by $p_0(x) = \sin(\pi.x)$ while the initial velocity is zero.

For the numerical simulations, the number of points of discretization in space on the interval $[0, L = 1]$ is given by $m = 20$ while the number of points of discretization in time on the interval $[0, T = 0.25]$ is equal to $N = 20$. Note that the discretization step in space is equal to $h = \frac{1}{21} = 0.0476$ while the discretization step in time is equal to $k = \frac{0.25}{20} = 0.0125$. Thus, for the explicit numerical scheme $c \frac{k}{h} = 0.2626 < 1$ and the C.F.L. condition is well verified. Moreover, $a \frac{k}{3} = 2 \cdot \frac{k}{3} = 0.0083 < 2$; thus, the explicit schemes will be stable.

Figure 2 represents, on the one hand, the curves of the exact solution and, on the other, the curves obtained by implementing the explicit and implicit numerical schemes.

All the numerical solutions are therefore in good agreement with the exact solution.

5.2 Damped wave equation equipped with mixed conditions.

We consider now the comparison of curves obtained by explicit and implicit numerical schemes with the exact solution when the boundary conditions are mixed. In this case, the exact solution of the problem is given by $p_{exact}(x, t) = x \cdot (x - 2) \cdot \exp(-b.t)$ with $b = 1$; the associated second member is given by $g(x, t) = (x \cdot (x - 2) \cdot (b \cdot (b - d) + a) - 2 \cdot c^2) \cdot \exp(-b.t)$ (with $c = 1$ and $a = d = 2$). Note that $p_{exact}(0, t) = 0$ and $\frac{\partial p(L=1, t)}{\partial x} = 0$ and that the

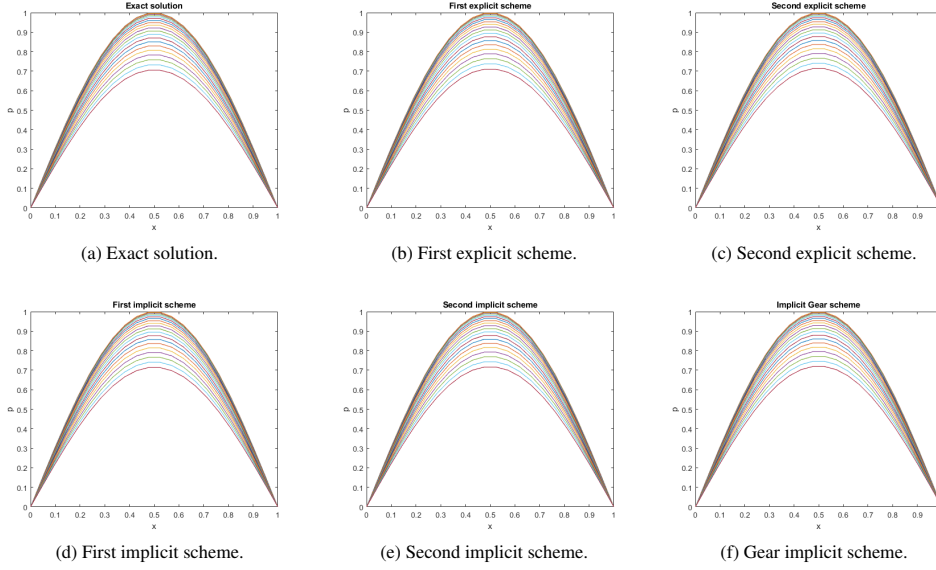


Fig. 2: Damped wave equation with homogeneous Dirichlet boundary conditions.

initial condition in time is given by $p_0(x) = x.(x - 2)$ while the initial velocity is equal to $\frac{\partial p(x,0)}{\partial x} = -b.x.(x - 2)$.

As above, we choose $m = 20$ and $N = 20$ and therefore, $h = 0.0476$ and $k = 0.0125$. For the explicit numerical scheme, the numerical stability conditions are verified.

Figure 3 represents, on the one hand, the curves representing the exact solution and, on the other hand, the curves obtained by implementing the explicit and implicit numerical schemes. **Remark 22** For example the choice of $m = 100$ produces an effect of instability in the numerical simulation. In this case, $c_{\frac{k}{h}} = \frac{0.0125}{0.00990} = 1.2625 > 1$ so that the C.F.L. condition is not satisfied. However, when $m = 50$ the explicit numerical schemes are numerically stable. Indeed, in this case, $c_{\frac{k}{h}} = \frac{0.0125}{0.01960} = 0.6375 < 1$ and the numerical explicit schemes run efficiently.

6 Conclusion.

In this study, the numerical analysis of various schemes for numerically solving the damped wave equation defined in one-, two- and three-dimensional domains was carried out. In particular, estimates of the truncation error of the schemes have been given and numerical stability has been studied. In the case of implicit schemes, an overview of algorithms for solving high-dimensional, sparse linear systems was given. In order to complete this study, numerical simulations have been carried out in simple situations of one-dimensional domains Ω .

In a further study, we plan to test our schemes by performing numerical simulations for problems defined in two-dimensional and three-dimensional domains Ω . In addition, we can also apply the proposed schemes to the numerical study of a piezoelectric delay problem modeled by a wave equation coupled to a damped wave equation.

the associated eigenvector. Let us write the k -th eigenvalue equation, so we obtain:

$$cv_{k-1} + (a - \lambda)v_k + bv_{k+1} = 0, k = 1, \dots, M,$$

with

$$v_0 = v_{M+1} = 0.$$

Let us determine the components v_k in the form $v_k = r^k$ where in general r is a complex number, so we obtain:

$$r^{k-1}(br^2 + (a - \lambda)r + c) = 0 \Rightarrow br^2 + (a - \lambda)r + c = 0,$$

since the components of V are different from zero (and consequently in general $r \neq 0$ also). Therefore, the component v_k is given by:

$$v_k = \alpha r_1^k + \beta r_2^k.$$

Note that $v_0 = 0$; this leads $v_0 = \alpha + \beta = 0$, i.e. $\alpha = -\beta$ and consequently:

$$v_k = \alpha(r_1^k - r_2^k).$$

In addition, we can also find the sum and the product of the second degree equation. Hence, we obtain:

$$r_1 + r_2 = \frac{\lambda - a}{b} \Rightarrow \lambda = a + b(r_1 + r_2),$$

and

$$r_1 r_2 = \frac{c}{b} \Rightarrow r_2 = \frac{1}{r_1} \frac{c}{b},$$

and consequently

$$v_k = \alpha \left(r_1^k - \frac{1}{r_1^k} \left(\frac{c}{b} \right)^k \right).$$

In particular for $k = M + 1$

$$v_{M+1} = 0 = \alpha \left(r_1^{M+1} - \frac{1}{r_1^{M+1}} \left(\frac{c}{b} \right)^{M+1} \right) \Rightarrow r_1^{2(M+1)} = \left(\frac{c}{b} \right)^{M+1},$$

and finally

$$r_1 = \pm \sqrt[2]{\frac{c}{b}} \exp(j \frac{k\pi}{M+1}) \text{ and } r_2 = \pm \sqrt[2]{\frac{c}{b}} \exp(-j \frac{k\pi}{M+1}), k = 1, \dots, 2(M+1),$$

where $j^2 = -1$. Thus,

$$r_1 + r_2 = \pm 2 \sqrt[2]{\frac{c}{b}} \cos\left(\frac{k\pi}{M+1}\right), k = 1, \dots, 2(M+1),$$

and

$$\lambda_k = a \pm 2\sqrt[2]{bc} \cos\left(\frac{k\pi}{M+1}\right), k = 1, \dots, 2(M+1).$$

Note that, we obtain the same values of λ_k , on one hand, for $k = 1, \dots, M$ and on the other hand, for $k = M+2, \dots, 2M+1$.

Furthermore, for $k = M+1$, let us find the components of V . Since, in this case, $r_1 = \pm\sqrt[2]{\frac{c}{b}} \exp(j\pi) = \mp\sqrt[2]{\frac{c}{b}}$ and $r_2 = \pm\sqrt[2]{\frac{c}{b}} \exp(j\pi) = \mp\sqrt[2]{\frac{c}{b}} = r_1$, then $v_{M+1} = \alpha(r_1^{M+1} - r_2^{M+1}) = 0$ and the index $k = M+1$ is not suitable.

In the same way, the value of $k = 2(M+1)$ leads to a value of $r_1 = r_2 = \pm\sqrt[2]{\frac{c}{b}}$ and $v_{2(M+1)}$ equals zero and consequently $k = 2(M+1)$ is not suitable.

Finally, the eigenvalues of the matrix \mathcal{A}_{1D} are given by:

$$\lambda_k = a \pm 2\sqrt[2]{bc} \cos\left(\frac{k\pi}{M+1}\right), k = 1, \dots, M.$$

The components of the eigenvector associated to the eigenvalue number k are given by:

$$v_l^{(k)} = \alpha(r_1^l - r_2^l) = \pm\alpha\sqrt[2]{\frac{c}{b}} (\exp(j\frac{lk\pi}{M+1}) - \exp(-j\frac{lk\pi}{M+1})),$$

and finally

$$v_l^{(k)} = \mp 2\alpha\sqrt[2]{\frac{c}{b}} \sin\left(\frac{lk\pi}{M+1}\right), k, l = 1, \dots, M.$$

Since, the eigenvector is defined to one multiplicative constant, we can normalize and write:

$$v_l^{(k)} = \sin\left(\frac{lk\pi}{M+1}\right), k, l = 1, \dots, M.$$

Remark 23 When $a = 2$ and $b = c = -1$, we find the eigenvalues of the Laplacian equipped with Dirichlet boundary conditions. Indeed, if we take the negative determination of the square root, we find:

$$\lambda_k = 2 - 2\cos\left(\frac{k\pi}{M+1}\right) = 4\sin^2\left(\frac{k\pi}{2(M+1)}\right), k = 1, \dots, M.$$

7.2 Case of one-dimensional Laplacian equipped with mixed boundary conditions.

Consider now the one-dimensional Laplacian equipped with mixed boundary conditions studied in section 3. As in previously done, we use a phantom point numbered $(M+2)$. This

The components of the eigenvectors associated to λ_k are given by:

$$v_i^k = \alpha(r_1^l - r_2^l) = \alpha\left(\exp\left(j\frac{(2k-1)l\pi}{2(M+1)}\right) - \exp\left(-j\frac{(2k-1)l\pi}{2(M+1)}\right)\right),$$

i.e.

$$v_i^k = 2j\alpha \sin\left(\frac{(2k-1)l\pi}{2(M+1)}\right), l, k = 1, \dots, M+1,$$

and, as before, we normalize by choosing $2.j.\alpha = 1$.

7.3 Case of the two-dimensional Laplacian equipped with Dirichlet boundary conditions.

Consider now the Poisson equation or more generally the convection-diffusion problem defined in a square domain and equipped with the Dirichlet boundary conditions. The discretization matrix is block-tridiagonal and is represented as follows:

$$\mathcal{A}_{2D} = \begin{pmatrix} \begin{array}{c|c|c} a & c & e \\ b & \ddots & \ddots \\ \ddots & \ddots & c \\ \hline & & b & a \\ \hline d & & & e \\ \ddots & & & \ddots \\ \hline & & d & e \\ \hline & & & d & a & c \\ & & & \ddots & b & \ddots \\ & & & & \ddots & c \\ & & & & & b & a \end{array} \end{pmatrix}, \quad (53)$$

with real coefficients. In the following, let us calculate the eigenelements of this matrix. The eigenvalue equation is written as:

$$dv_{k,l-1} + bv_{k-1,l} + av_{k,l} + cv_{k+1,l} + ev_{k,l+1} = \lambda v_{k,l}, k = 1, \dots, N, l = 1, \dots, M,$$

where, for example, l is the block number, and k the component number in block number l ; M denotes the number of blocks and N the size of each block. By considering the Dirichlet boundary conditions, which can always be reduced to the homogeneous case since the problem is linear, we therefore write, for $k = 1, \dots, N, l = 1, \dots, M$:

$$\begin{cases} (dv_{k,l-1} + \frac{1}{2}(a - \lambda)v_{k,l} + ev_{k,l+1}) + (bv_{k-1,l} + \frac{1}{2}(a - \lambda)v_{k,l} + cv_{k+1,l}) = 0, \\ v_{0,l} = v_{N+1,l} = v_{k,0} = v_{k,M+1} = 0. \end{cases}$$

In order to return to the one-dimensional case, we consider a variable separation by searching $v_{k,l}$ in the form $v_{k,l} = f(k)g(l)$ which amounts to separating k and l . Thus, for $k = 1, \dots, N, l = 1, \dots, M$, we have:

$$\begin{cases} f(k)(dg(l-1)) + \frac{1}{2}(a-\lambda)g(l) + eg(l+1) + g(l)(bf(k-1) + \frac{1}{2}(a-\lambda)f(k) + cf(k+1)) = 0, \\ f(0) = f(N+1) = g(0) = g(M+1) = 0. \end{cases}$$

Since $v_{k,l}$ is generally not equal to zero, we divide the two members by $f(k).g(l)$. Thus, we obtain for all k and all l :

$$\frac{dg(l-1) + eg(l+1)}{g(l)} + \frac{1}{2}(a-\lambda) = -\frac{bf(k-1) + cf(k+1)}{f(k)} - \frac{1}{2}(a-\lambda) = C,$$

where C is a constant. Finally, we obtain:

$$\begin{cases} bf(k-1) + \frac{1}{2}(a-\lambda + 2C)f(k) + cf(k+1) = 0, k = 1, \dots, N, \\ f(0) = f(N+1) = 0, \end{cases}$$

and

$$\begin{cases} dg(l-1) + \frac{1}{2}(a-\lambda - 2C)g(l) + eg(l+1) = 0, l = 1, \dots, M, \\ g(0) = g(M+1) = 0, \end{cases}$$

and we are therefore reduced to two decoupled problems identical to those previously considered in the one-dimensional case. We therefore look for $f(k)$ in the form $f(k) = s^k$ and for $g(l)$ in the form $g(l) = r^l$. As previously, since the components of the eigenvectors are generally nonzero, we have to solve the following two second degree equations:

$$\begin{cases} cs^2 + \frac{1}{2}(a-\lambda + 2C)s + b = 0, \\ er^2 + \frac{1}{2}(a-\lambda - 2C)r + d = 0, \end{cases}$$

each of which admits two roots; so we have:

$$f(k) = \alpha s_1^k + \beta s_2^k \text{ and } g(l) = \alpha' r_1^l + \beta' r_2^l.$$

By taking into account the boundary conditions, we obtain:

$$f(0) = 0 = \alpha + \beta \Rightarrow \beta = -\alpha \text{ and } g(0) = 0 = \alpha' + \beta' \Rightarrow \beta' = -\alpha'.$$

Remark 24 *In the following, we exclude the case of a component of V equal to zero, which in fact implies excluding the cases where $s_1 = s_2$ and $r_1 = r_2$.*

Furthermore, by taking into account the other boundary condition, we obtain:

$$f(N+1) = 0 = \alpha(s_1^{N+1} - s_2^{N+1}) \text{ and } g(M+1) = 0 = \alpha'(r_1^{M+1} - r_2^{M+1}),$$

which implies

$$s_1^{N+1} = s_2^{N+1} \text{ and } r_1^{M+1} = r_2^{M+1}.$$

In addition,

$$s_1.s_2 = \frac{b}{c} \text{ and } s_1 + s_2 = \frac{\lambda - a - 2C}{c} \text{ while } r_1.r_2 = \frac{d}{e} \text{ and } r_1 + r_2 = \frac{\lambda - a + 2C}{e}.$$

As above, by handling index bounds, as in the one-dimensional case, we deduce for $t = 1, \dots, N$, that:

$$s_1^{2(N+1)} = \left(\frac{b}{c}\right)^{N+1} \Rightarrow s_1 = \pm \sqrt[2]{\frac{b}{c}} \exp(j \frac{t\pi}{N+1}) \text{ and } s_2 = \pm \sqrt[2]{\frac{b}{c}} \exp(-j \frac{t\pi}{N+1}),$$

and for $w = 1, \dots, M$,

$$r_1^{2(M+1)} = \left(\frac{d}{e}\right)^{M+1} \Rightarrow r_1 = \pm \sqrt[2]{\frac{d}{e}} \exp(j \frac{w\pi}{M+1}) \text{ and } r_2 = \pm \sqrt[2]{\frac{d}{e}} \exp(-j \frac{w\pi}{M+1}).$$

Moreover, we have:

$$\lambda - a - 2C = \pm 4 \sqrt[2]{bc} \cos\left(\frac{t\pi}{N+1}\right) \text{ and } \lambda - a + 2C = \pm 4 \sqrt[2]{de} \cos\left(\frac{w\pi}{M+1}\right),$$

and by adding these two relations, we deduce the expression of the eigenvalues of \mathcal{A}_{2D} :

$$\lambda_{tw} = a \pm 2 \sqrt[2]{bc} \cos\left(\frac{t\pi}{N+1}\right) \pm 2 \sqrt[2]{de} \cos\left(\frac{w\pi}{M+1}\right), t = 1, \dots, N, w = 1, \dots, M.$$

We also deduce the components of the eigenvectors V which, after normalization, are given by:

$$v_{k,l}^{t,w} = \sin\left(\frac{tk\pi}{N+1}\right) \sin\left(\frac{wl\pi}{M+1}\right), t, k = 1, \dots, N, w, l = 1, \dots, M.$$

Remark 25 *In the case of the 2D-Poisson equation defined in a square with Dirichlet boundary conditions, we have $a = 4$ and $b = c = d = e = -1$; then, by taking the negative determination of the square root, the eigenvalues of the matrix \mathcal{A}_{2D} are given by:*

$$\lambda_{tw} = 4\left(\sin^2\left(\frac{t\pi}{2(N+1)}\right) + \sin^2\left(\frac{w\pi}{2(M+1)}\right)\right), t = 1, \dots, N, w = 1, \dots, M.$$

7.4 Case of the three-dimensional Laplacian equipped with Dirichlet boundary conditions.

Finally, let us consider the Poisson equation or also the convection-diffusion problem defined in a cubic domain and equipped with Dirichlet boundary condition. The discretization matrix is a heptadiagonal sparse matrix with a block-tridiagonal structure, shown below:

$$\mathcal{A}_{3D} = \begin{pmatrix} \mathcal{A}_{2D} & p^{\mathbb{I}} & & & & & \\ q^{\mathbb{I}} & \mathcal{A}_{2D} & p^{\mathbb{I}} & & & & \\ & q^{\mathbb{I}} & \mathcal{A}_{2D} & p^{\mathbb{I}} & & & \\ & & q^{\mathbb{I}} & \mathcal{A}_{2D} & p^{\mathbb{I}} & & \\ & & & q^{\mathbb{I}} & \mathcal{A}_{2D} & & \\ & & & & q^{\mathbb{I}} & \mathcal{A}_{2D} & \\ & & & & & q^{\mathbb{I}} & \mathcal{A}_{2D} \end{pmatrix}, \quad (54)$$

where \mathcal{A}_{2D} represents the discretization matrix on in each plane and \mathbb{I} is the identity matrix; the entries of \mathcal{A}_{2D} are denoted in the same way as in the previous subsection. For $k = 1, \dots, N, l = 1, \dots, M, i = 1, \dots, Q$ the eigenvalue equation of \mathcal{A}_{3D} is written as follows:

$$qv_{k,l,i-1} + dv_{k,l-1,i} + bv_{k-1,l,i} + av_{k,l,i} + cv_{k+1,l,i} + ev_{k,l+1,i} + pv_{k,l,i+1} = \lambda v_{k,l,i},$$

where N represent the tridiagonal block size, M the number of tridiagonal blocks, and Q the number of planes. In the same way as in the two-dimensional case, we search $v_{k,l,i}$ in the following form:

$$v_{k,l,i} = f(k)g(l)h(i).$$

By injecting this new expression of $v_{k,l,i}$ into the eigenvalue equation, and after regrouping the terms to take into account of the variable separation, we obtain for $k = 1, \dots, N, l = 1, \dots, M, i = 1, \dots, Q$:

$$\begin{aligned} f(k)g(l)(qh(i-1)) + \frac{1}{3}(a-\lambda)h(i) + ph(i+1) \\ + g(l)h(i)(bf(k-1) + \frac{1}{3}(a-\lambda)f(k) + cf(k+1)), \\ + f(k)h(i)(dg(l-1) + \frac{1}{3}(a-\lambda)g(l) + eg(l+1)) = 0. \end{aligned}$$

Since $v_{k,l,i} \neq 0$ for all k, l, i let us divide by $f(k)g(l)h(i)$ which gives rise to:

$$\begin{aligned} \frac{qh(i-1) + ph(i+1)}{h(i)} + \frac{1}{3}(a-\lambda) \\ = -\frac{bf(k-1) + cf(k+1)}{f(k)} - \frac{dg(l-1) + eg(l+1)}{g(l)} - 2\frac{1}{3}(a-\lambda) = R \end{aligned}$$

Thus, the left-hand side depending only on $h(i)$ is equal to a constant R and we have:

$$qh(i-1) + \frac{1}{3}(a-\lambda-3R)h(i) + ph(i+1) = 0, i = 1, \dots, Q.$$

For the right-hand side, we have:

$$-\frac{bf(k-1) + cf(k+1)}{f(k)} - \frac{1}{3}(a-\lambda) = R + \frac{dg(l-1) + eg(l+1)}{g(l)} + \frac{1}{3}(a-\lambda) = S.$$

From this previous relation, we can further decouple the relations between $f(k)$ and $g(l)$ to obtain the following relations:

$$bf(k-1) + \frac{1}{3}(a-\lambda+3S)f(k) + cf(k+1) = 0, k = 1, \dots, N,$$

and

$$dg(l-1) + \frac{1}{3}(a-\lambda+3R-3S)g(l) + eg(l+1) = 0, l = 1, \dots, M.$$

As in the two-dimensional case, we obtain three decoupled relations satisfied by $h(i)$, $f(k)$ and $g(l)$. We then operate as in the one-dimensional case, looking for $h(i)$ in the form $h(i) = t^i$ (respectively $f(k) = s^k$ and $g(l) = r^l$). This leads to the three following equations of second degree:

$$\begin{cases} pt^2 + \frac{1}{3}(a - \lambda - 3R)t + q = 0, \\ cs^2 + \frac{1}{3}(a - \lambda + 3S)t + b = 0, \\ er^2 + \frac{1}{3}(a - \lambda + 3R - 3S)r + d = 0, \end{cases}$$

where R and S are two constants. Due to the linear nature of the equation, we can always consider the boundary conditions to be zero, i.e. $f(0) = g(0) = h(0) = 0$, which gives:

$$f(k) = \alpha(s_1^k - s_2^k); g(l) = \alpha'(r_1^l - r_2^l) \text{ and } h(i) = \alpha''(t_1^i - t_2^i).$$

The cases $s_1 = s_2$, $r_1 = r_2$ and $t_1 = t_2$ are excluded since the components of the eigenvectors are different to zero. The other boundary condition leads to:

$$s_1^{N+1} = s_2^{N+1}, r_1^{M+1} = r_2^{M+1} \text{ and } t_1^{Q+1} = t_2^{Q+1}.$$

From the expressions of the equations of the second degree, we obtain the sum and the product of the roots, i.e.

$$\begin{cases} r_1.r_2 = \frac{d}{e} \text{ and } r_1 + r_2 = -\frac{1}{3}\frac{a-\lambda+3R-3S}{e}, \\ s_1.s_2 = \frac{b}{c} \text{ and } s_1 + s_2 = -\frac{1}{3}\frac{a-\lambda+3S}{c}, \\ t_1.t_2 = \frac{q}{p} \text{ and } t_1 + t_2 = -\frac{1}{3}\frac{a-\lambda-3R}{p}. \end{cases}$$

In the same way, than for the one-dimensional and two-dimensional cases, we get:

$$\begin{cases} r_1 = \pm \sqrt[2]{\frac{d}{e}} \exp(j \frac{w\pi}{M+1}) \text{ and } r_2 = \pm \sqrt[2]{\frac{d}{e}} \exp(-j \frac{w\pi}{M+1}), w = 1, \dots, M, \\ s_1 = \pm \sqrt[2]{\frac{b}{c}} \exp(j \frac{v\pi}{N+1}) \text{ and } s_2 = \pm \sqrt[2]{\frac{b}{c}} \exp(-j \frac{v\pi}{N+1}), v = 1, \dots, N, \\ t_1 = \pm \sqrt[2]{\frac{q}{p}} \exp(j \frac{u\pi}{Q+1}), t_2 = \pm \sqrt[2]{\frac{q}{p}} \exp(-j \frac{u\pi}{Q+1}), u = 1, \dots, Q. \end{cases}$$

From these relations, we deduce that:

$$\begin{cases} \frac{\lambda}{3} = \frac{a}{3} \pm 2\sqrt[2]{bc} \cos(\frac{lw\pi}{M+1}) + R - S, \\ \frac{\lambda}{3} = \frac{a}{3} \pm 2\sqrt[2]{de} \cos(\frac{tk\pi}{N+1}) + S, \\ \frac{\lambda}{3} = \frac{a}{3} \pm 2\sqrt[2]{pq} \cos(\frac{ui\pi}{Q+1}) - R. \end{cases}$$

After simplification, by summing these three previous relations, this gives the eigenvalues of matrix \mathcal{A}_{3D} , i.e. for $l = 1, \dots, M$, $k = 1, \dots, N$ and $i = 1, \dots, Q$:

$$\lambda_{kli} = a \pm 2\sqrt[2]{bc} \cos(\frac{l\pi}{M+1}) \pm 2\sqrt[2]{de} \cos(\frac{k\pi}{N+1}) \pm 2\sqrt[2]{pq} \cos(\frac{i\pi}{Q+1}).$$

For the components of the associated eigenvectors, after normalization, we obtain, for $l, w = 1, \dots, M, t, k = 1, \dots, N$ and $u, i = 1, \dots, Q$:

$$v_{kli}^{t,w,u} = \sin\left(\frac{lw\pi}{M+1}\right) \sin\left(\frac{kt\pi}{N+1}\right) \sin\left(\frac{i\pi}{Q+1}\right).$$

Remark 26 *If we consider the discretization matrix of the Laplacian equipped with Dirichlet boundary conditions, since $a = 6$ and $b = c = d = e = p = q = -1$, we get for $t = 1, \dots, Q, u = 1, \dots, N$ and $w = 1, \dots, M$, the values of the eigenvalues of \mathcal{A}_{3D} :*

$$\lambda_{t,w,u} = 4\left(\sin^2\left(\frac{t\pi}{2(N+1)}\right) + \sin^2\left(\frac{w\pi}{2(M+1)}\right) + \sin^2\left(\frac{u\pi}{2(M+1)}\right)\right).$$

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