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# Modern vs. classical structures of opposition: A discussion

Didier Dubois, Henri Prade and Agnès Rico

**Abstract.** The aim of this work is to revisit the proposal made by Dag Westerståhl a decade ago when he provided a modern reading of the traditional square of opposition and of related structures. We propose a formalization of this modern view and contrast it with the classical one. We discuss what may be a modern hexagon of opposition and a modern cube, and show their interest in particular for relating quantitative expressions.

**Mathematics Subject Classification (2010).** Primary 68T30; Secondary 03A05, 03B05, 68T37.

**Keywords.** square of opposition, hexagon of opposition, cube of opposition.

## 1. Introduction

The traditional square of opposition [32] is a basic diagram that exhibits the logical interaction between four related statements. This structure can be encountered in many knowledge representation settings [11]. Its underlying logical structure is well understood [5]; it involves negations, implications and mutual exclusions.

A decade ago, Westerståhl [40], a philosopher and a logician interested in linguistic quantifiers has identified a slightly different structure on a number of examples, where implications and mutual exclusions are replaced by simple duality relations. He called such a structure a “modern square”.

The goal of this paper is to formalize the concept of modern square, to provide generic examples, and to discuss its differences with traditional squares. In particular, we look for modern squares that are not traditional ones, and we show that some of them are “reversible” traditional squares, i.e., depending on the value of a parameter, the modern square is a traditional square, or a traditional square upside down. We investigate the modern counterparts of other structures of opposition related to the square, such as hexagons and cubes. We also show the interest of modern structures for relating quantitative expressions, which cannot be considered as true or false but are a matter of degree.

This paper develops a discussion originated in [18]. It complements and deepens an investigation started in [17] and illustrated there on a family of aggregation operators (Sugeno integrals).

## 2. Squares of opposition

This section starts with a reminder of the classical square of opposition. The second subsection presents the modern square of opposition and its links with the classical one. The last section is devoted to examples, such as for instance, parameterized squares.

### 2.1. The classical square

The traditional square of opposition [32] is built with universally and existentially quantified statements in the following way. Consider a statement (**A**) of the form “all  $x$ ’s are  $B$ ’s”, which is negated by the statement (**O**) “at least one  $x$  is not a  $B$ ”, together with the statement (**E**) “no  $x$  is a  $B$ ”, which is clearly in even stronger opposition (than **O**) to the first statement (**A**). These three statements, together with the negation of the last statement, namely (**I**) “at least one  $x$  is a  $B$ ” can be displayed on a square whose vertices are traditionally denoted by the letters **A**, **I** (Affirmative half: from Latin “Affirmo”) and **E**, **O** (nEgative half: from Latin “nEgo”), as pictured in Figure 1 (where  $\bar{B}$  stands for “not  $B$ ”).

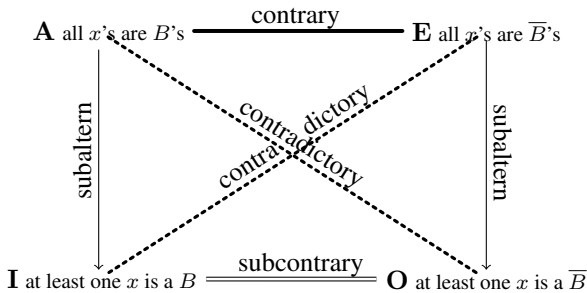


FIGURE 1. Traditional square of opposition

We shall refer to this square as the *classical* square of opposition. As can be checked, noticeable relations hold in the square:

- (i) **A** and **O** (resp. **E** and **I**) are the negation of each other;
- (ii) **A** entails **I**, and **E** entails **O** (it is assumed that there is at least one  $x \in B$  and one  $x' \notin B$  thus avoiding existential import problems);
- (iii) **A** and **E** cannot be true together, but both may be false;
- (iv) **I** and **O** cannot be false together, but both may be true.

Using the above relations, there are three equivalent options for defining a *formal* square of opposition with independent conditions, thus laying bare the logical structure of the square of opposition [5]. We choose one of the options for the following definition, and give the other equivalent options right after.

**Definition 1.** *AEOI is a square of opposition if the two following conditions hold:*

- (i) *A and O are contradictories:  $A \equiv \neg O$ ,  $E \equiv \neg I$  (diagonal link  $A - O$ ).*
- (iii) *A and E are mutually exclusive (i.e., contraries):  $\neg A \vee \neg E$  holds.*

Equivalent definitions are obtained when replacing condition (iii) by one of the following:

- (ii) Both implication relations  $A \rightarrow I$  and  $E \rightarrow O$  hold (vertical arrows).
- (iv) I and O are subcontraries (cover all situations):  $I \vee O$  holds.

**2.2. The modern square**

D. Westerståhl [40] has proposed a so-called “modern” reading of the square of opposition **AEOI** (see also [27]). This reading is quite different from the classical one where logical constraints are supposed to hold between the vertices of the square (see Def. 1). In the modern reading, **A** and **E**, as well as **I** and **O**, exchange through an involutive *internal negation*, and **A** and **I** (resp. **E** and **O**) are dual to each other, where *duality* is obtained by composing the internal negation with the (involutive) *external negation* that is assumed to hold between diagonally opposite vertices.

So, in a modern square we keep property (i) expressing the effect of the external negation and we add the use of an involutive inner negation between **A** and **E**. The vertices of the square are associated with statements that are supposed to be true or false. Such statements are supposed to be of the form  $S(B)$  where  $B$  denotes another statement of particular interest. Thus  $B$  may refer to a variable in a logical proposition, to a subset of items, to a number in a comparative statement, such as, for instance, respectively  $B = q$  in  $S(q) = ‘p \wedge q’$ ,  $S(B) = ‘\text{all } x\text{'s are } B\text{'s}’$ , or  $S(B) = ‘\theta > \alpha’$  (then  $B$  is a number  $\theta$  in  $[0, 1]$ , for instance). We denote by  $n(B)$  the internal negation of  $B$ , with  $n(n(B)) = B$  (involution); for instance, if  $B = \theta$ , then  $n(B) = 1 - \theta$ . Since  $S(B)$  is a Boolean statement, we denote by  $\neg S(B)$  the external negation of this statement. Obviously  $\neg\neg S(B) = S(B)$ .

Hence a modern square, as pictured in Figure 2, is defined as follows:

**Definition 2.** *A modern square AEOI, where A is associated with  $S(B)$ , is such that*

- *A and O, as well as E and I, exchange via an involutive external negation  $\neg$ .*
- *A and E exchange via an involutive internal negation  $n$ , namely E is associated with  $S(n(B))$ .*

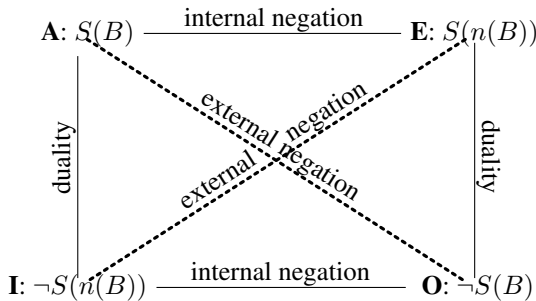


FIGURE 2. A modern square of opposition

Note that **I** and **O** also exchange via the internal negation. Besides, a duality between **A** and **I** and between **E** and **O** is obtained by composing the internal and external negations. In the remainder of this paper, *the word ‘duality’ is restricted to this precise use*. Namely, the simple interplay between two expressions by means of one negation will be identified as such and not called ‘duality’.

It is worth noticing that if we add the condition that **A** and **E** cannot be true together to Definition 2 of the modern square, we retrieve the definition of the classical square.

It is also possible to define a modern structure without insisting on the “internal” nature of  $n$ , just considering that we have two involutive “negations”,  $\neg$  and  $n$ , that apply to a statement  $S$ , as in Figure 3. But in this case, we should add the requirement  $\neg n(S) = n(\neg S)$  in order to ensure that relations between **A** and **E** and between **I** and **O** are the same.

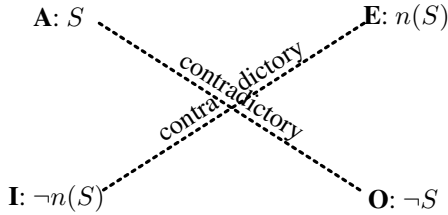


FIGURE 3. Abstracting the core of the square of opposition

### 2.3. Some examples of classical squares and modern squares

Let us first notice that the traditional square in Figure 1 may have a modern reading if we forget the distinction between “contrary” and “sub-contrary” and if we forget the “subaltern” relations. Indeed for going from **A** to **E** and from **I** to **O** we apply a negation  $n$  by taking the complement of  $B$ , so that “all” and “at least one” are dual of each other.

Indeed some modern squares are classical, as in the two examples below.

**Example 1.** We consider two sets  $B$  and  $C$  included in a universe  $\mathcal{X}$ .  $A$  is associated with the statement  $S(B) = 'C \subseteq B'$  and  $n(B) = \overline{B}$ . The square **AEOI** such that  
 $A : C \subseteq B$        $E : C \subseteq \overline{B}$   
 $I : C \cap B \neq \emptyset$        $O : C \cap \overline{B} \neq \emptyset$   
 is classical and modern, as can be easily checked.

**Example 2.** We consider a partition  $C_i$   $i \in \{1, \dots, N\}$  on an ordered, finite universe  $\mathcal{X}$ . The internal negation is the antonym, denoted by  $\text{ant}$ , and defined, for all  $i \in \{1, \dots, N\}$ , by  $\text{ant}(C_i) = C_{N-i+1}$  and for any subset  $\mathbb{C} \subseteq \{C_1, \dots, C_N\}$  we have  $\text{ant}(\mathbb{C}) = \{\text{ant}(C_i) : C_i \in \mathbb{C}\}$ . The external negation is the set complement  $\text{not } \mathbb{C} = \{C_i \notin \mathbb{C}\}$ . Hence the modern square **AEOI**

$$\begin{array}{ll} A : x \text{ is } C_i & E : x \text{ is } \text{ant}(C_i) \\ I : x \text{ is not } \text{ant}(C_i) & O : x \text{ is not } C_i \end{array}$$

is classical. Indeed,  $C_i \notin \text{ant}(C_i)$  which entails an implication relation between **A** and **I**.

For instance, we may consider the set  $Y$  of young people, the set  $M$  of middle-aged people and the set  $S$  of senior people. This yields the following square (where  $Y = \text{ant}(S)$ ,  $M = \text{ant}(M)$ , so that  $M$  or  $S = \text{ant}(Y \text{ or } M)$ ).

$A : x \text{ is } Y$              $E : x \text{ is } S$   
 $I : x \text{ is } Y \text{ or } M$      $O : x \text{ is } M \text{ or } S$ .

Another instance is with  $\mathcal{X} = \{\text{black, grey, white}\}$ , then if  $A$  is “ $x$  is black” then  $I$  is “ $x$  is black or grey”,  $E$  is “ $x$  is white”, and  $O$  is “ $x$  is grey or white”, where  $\text{white or grey} = \text{ant}(\text{grey or black})$ .

But some modern squares are not classical. We first consider an example where the square may become classical after the addition of particular constraints.

**Example 3.** On a universe  $\mathcal{X}$ , suppose two strict and non-empty subsets  $B$  and  $C$  of  $\mathcal{X}$ . The external negation of a set  $B$  is its complement  $\bar{B}$ . The internal negation of  $B$  (resp.  $C$ ) is defined as the complement of  $C$  (resp.  $B$ ). See Figure 4.

We assume neither  $B \subseteq C$  nor  $C \subseteq B$  (so there is no implication from  $A$  to  $I$ , nor from  $I$  to  $A$ ). So the square  $AEOI$  is modern but not classical.

However, if we add the condition  $B \cap C = \emptyset$  (or  $B \cup C = \mathcal{X}$ ), the modern square becomes a classical one by rotating it 90 clockwise since if  $C \cap B = \emptyset$  then  $C \subseteq \bar{B}$ , or turning it 90 counter-clockwise (in the case of the second condition).

Besides, note that in case we allow for  $B \subseteq C$  or  $C \subseteq B$ , the square  $AEOI$  or the square  $IOEA$  becomes classical.

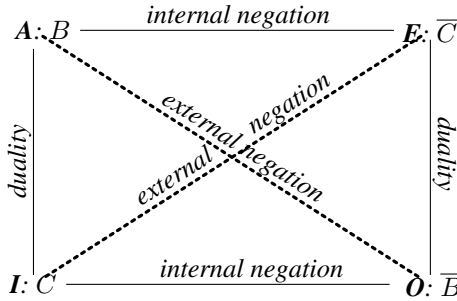


FIGURE 4. A modern square of opposition induced by two subsets non-included in each other

This situation of subsets non-included in each other is encountered for instance in the case of  $\{0, 1\}$ -valued capacities. Let  $\mu$  be a Boolean capacity (see, e.g., [8]) on a finite set  $[N] = \{1, \dots, N\}$ .  $\mu : 2^{[N]} \rightarrow \{0, 1\}$  is an increasing set function such that  $\mu(\emptyset) = 0$  and  $\mu(\{1, \dots, N\}) = 1$ . Capacities are characterised by their focal sets, defined as follows.

First, the Möbius transform of the capacity  $\mu$  is a mapping  $\mu_{\#} : 2^{[N]} \rightarrow \{0, 1\}$  such that

- i)  $\mu_{\#}(A) = \mu(A)$  if for all  $B \subset A$  we have  $\mu(B) < \mu(A)$ , and
- ii)  $\mu_{\#}(A) = 0$  otherwise.

A focal set of  $\mu$  is a set  $A$  such that  $\mu_{\#}(A) > 0$ . It is clear that two focal sets  $B$  and

$C$  are such that  $B \not\subseteq C$  and  $C \not\subseteq B$ . So, these two focal sets of a Boolean capacity can be put on a modern square as in Figure 4 where neither  $B \subseteq C$  nor  $C \subseteq B$  hold.

We now provide another example of modern square that is non classical, using comparative relations.

**Example 4.** We take  $S(\theta) = ‘\theta < \alpha’$  where  $\theta$  and  $\alpha$  are numbers in  $[0, 1]$ .  $\alpha$  is taken as a reference in the comparison. The external negation is the standard negation and  $n$  is the complement to 1.

$$\mathbf{A} : \theta < \alpha \qquad \mathbf{E} : 1 - \theta < \alpha$$

$$\mathbf{I} : 1 - \theta \geq \alpha \qquad \mathbf{O} : \theta \geq \alpha.$$

This square is modern but there are no implication relations between  $\mathbf{A}$  and  $\mathbf{I}$ . We have neither  $\mathbf{A} \rightarrow \mathbf{I}$ , nor  $\mathbf{I} \rightarrow \mathbf{A}$  in general. However if  $\alpha \leq 0.5$ , then  $\mathbf{A}$  implies  $\mathbf{I}$ ; if  $\alpha > 0.5$  the converse holds. In these two cases  $\mathbf{AEOI}$  and  $\mathbf{IOEA}$  are respectively classical.

The latter example suggests the notion of parameterized square, where depending on the value of the parameter, different readings of the original modern square may lead to different classical squares.

**2.4. A subclass of modern squares: parameterized squares**

We now consider squares induced by statements of the form  $S(B) = ‘g(B) \geq \alpha’$ , where  $g(B)$  and  $\alpha$  are numbers, and  $B$  is a subset of items.<sup>1</sup> So important examples intended by the symbol  $g$  are set functions such as cardinality, probability and other uncertainty measures such as belief or possibility functions.

**Definition 3.** A parameterized modern square of opposition  $\mathbf{AEIO}_\alpha$  is a modern square of opposition where each vertex is associated to a parameterized expression depending on a parameter  $\alpha$ .

Such a parameterized square is depicted in Figure 5. It is only a modern square since in general, there no implication relation between  $\mathbf{A}$  and  $\mathbf{I}$ .

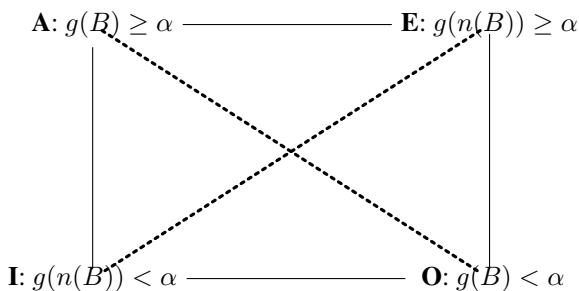


FIGURE 5. A parameterized modern square

<sup>1</sup>We do not consider other forms of parameterization in this article.

Note that if in Figure 5, we consider set-valued expressions such as  $\{B \mid g(B) \geq \alpha\}$  in place of the corresponding inequality, we still have a modern square, since sets inherit from the logical relations between the conditions that define them (and the external negation is now the set complementation). This square may become classical, e.g., if some additional conditions are assumed such as  $g(B) \geq \alpha$  implies  $g(n(B)) < \alpha$ ; for instance  $n$  is the set complementation and  $g$  is a probability measure.

We now discuss different instances of parameterized squares.

**Example 5.** Let us consider a probability measure  $P$  and a level  $\alpha \in [0, 1]$ . In this case  $B$  is a set, the negation  $n$  is the set complement, and  $S(B) = P(B) \geq \alpha$ .

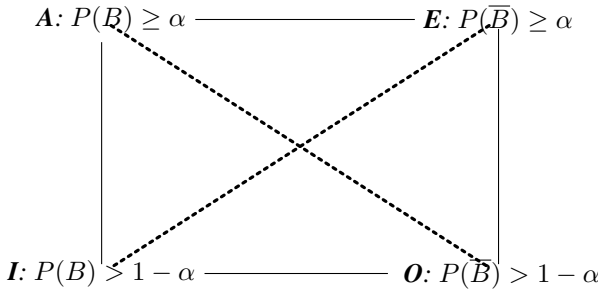


FIGURE 6. A modern square induced by a probability bound

We get the square on Figure 6. It is reversible in the following sense

- if  $\alpha > 0.5$  then the square **AEOI** is classical.
- if  $\alpha \leq 0.5$  then **IOEA** is classical.

The previous square straightforwardly generalizes to conditional probabilities starting with **A** associated with  $P(B|A) \geq \alpha$ . This conditional probability square extends previous examples by Westerståhl [40] who uses relative cardinalities and proportional quantifiers. Pfeifer and Sanfilippo [30] have also discussed this probabilistic square of opposition.

Another reversible square example can be observed for the square “at least  $k$ ” also considered by Westerståhl [40], and presented in the next example.

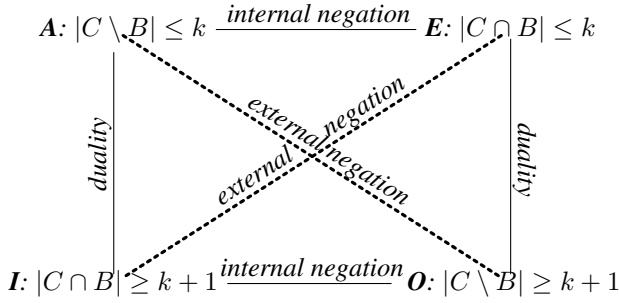
**Example 6.** Let  $B$  and  $C$  two subsets of a universe  $\mathcal{X}$ . Since  $C \setminus B = C \cap \bar{B}$ , it is easy to see that the square **AEOI** in Figure 7 is a modern square. We have  $|C \cap B| + |C \cap \bar{B}| = |C|$  so it can be checked that

- If  $|C| > 2k$  then  $|C \setminus B| \leq k$  implies  $|C \cap B| \geq k + 1$ .
- If  $|C| \leq 2k$  then  $|C \cap B| \geq k + 1$  implies  $|C \setminus B| \leq k$ .

So, the square “at least  $k$ ” **AEOI** is a classical square if and only if  $|C| > 2k$  is assumed. Otherwise, if  $|C| \leq 2k$ , the square **IOEA** is classical.

We conclude this subsection with an example of parameterized square where reversibility is lost.



FIGURE 7. Westerståhl's square "at least  $k$ "

**Example 7.** We consider a universe  $\mathcal{X}$  and a mass function defined on it, namely  $m : 2^{\mathcal{X}} \rightarrow [0, 1]$  with  $m(\emptyset) = 0$ , such that  $\sum_{B \subseteq \mathcal{X}} m(B) = 1$ . The associated belief function [37] is  $bel(B) = \sum_{C|C \subseteq B} m(C)$  and the plausibility function is  $pl(B) = \sum_{C|C \cap B \neq \emptyset} m(C)$ . We have  $bel(B) \leq pl(B)$  for all  $B$ 's. These set functions are conjugate:  $pl(B) = 1 - bel(\overline{B})$ ; thus  $bel(B) < \alpha$  is equivalent to  $pl(\overline{B}) > 1 - \alpha$ .

Considering  $\alpha \in [0, 1]$  we have the following modern square:

$$\begin{array}{ll} \mathbf{A} : bel(B) \geq \alpha, & \mathbf{E} : bel(\overline{B}) \geq \alpha \\ \mathbf{I} : pl(B) > 1 - \alpha & \mathbf{O} : pl(\overline{B}) > 1 - \alpha. \end{array}$$

There are two cases:

- If  $\alpha > 0.5$  this modern square becomes classical. Indeed, it implies  $1 - \alpha < \alpha$ , and then we have  $1 - \alpha < \alpha \leq bel(B) \leq pl(B)$ , which means that **A** implies **I** and **E** implies **O**.
- If  $\alpha \leq 1 - \alpha$ , i.e.,  $\alpha \leq 0.5$ , nothing can be said regarding  $pl(B)$  beyond  $\alpha \leq bel(B) \leq pl(B)$ , and we can prove neither that **A** implies **I** nor that **I** implies **A**. So the square is not reversible.

This situation is related to the fact that we are no longer working with additive set functions (indeed  $bel(B) + bel(\overline{B}) \leq 1$ ).

## 2.5. Graded square

Another square of opposition has been already introduced for belief functions [11]. However it was done in the extended setting of *graded* squares, where vertices are associated with expressions whose value is a matter of degree between 0 and 1. This means that, e.g., for vertex **A** we have just  $bel(B)$  instead of  $bel(B) \geq \alpha$ . This gives birth to the following square.

**Example 8.**

$$\begin{array}{ll} \mathbf{A} : bel(B) & \mathbf{E} : bel(\overline{B}) \\ \mathbf{I} : pl(B) & \mathbf{O} : pl(\overline{B}). \end{array}$$

In this graded square [11, 16], contradictions between **A** and **O** (resp. **E** and **I**) is expressed by  $bel(B) = 1 - pl(\overline{B})$  (resp.  $pl(B) = 1 - bel(\overline{B})$ ), contrariety between **A** and **E** is expressed by  $bel(B) + bel(\overline{B}) \leq 1$ , sub-contrariety between **I**

and  $\mathbf{O}$  by  $pl(B) + pl(\bar{B}) \geq 1$ , and vertical implications from  $\mathbf{A}$  to  $\mathbf{I}$  and from  $\mathbf{E}$  to  $\mathbf{O}$  by  $bel(B) \leq pl(B)$  and  $bel(\bar{B}) \leq pl(\bar{B})$  respectively.<sup>2</sup>

Examples 7 and 8 show that (i) building squares from Boolean statements comparing a numerical quantity to a threshold, or (ii) building graded squares directly using the numerical quantities, may not lead to equivalent structures of opposition.

The situation of Example 8 can be generalized to a general monotonically increasing set function (i.e., a capacity)  $\mu$  in place of  $bel$  in the above square. We shall obtain a modern square in the general case.

**Example 9.** Let  $[N] = \{1, \dots, N\}$  be a finite set and let  $\mu$  be a qualitative capacity [8] defined on its power set, i.e., a (non decreasing) mapping from  $2^{[N]}$  to a totally ordered set  $L$  equipped with an involutive order reversing map denoted by  $1 - (\cdot)$  (e.g.,  $L = [0, 1]$ ). We define the conjugate capacity  $\mu^c(B) = 1 - \mu(\bar{B})$ . We obtain the square in Figure 8.

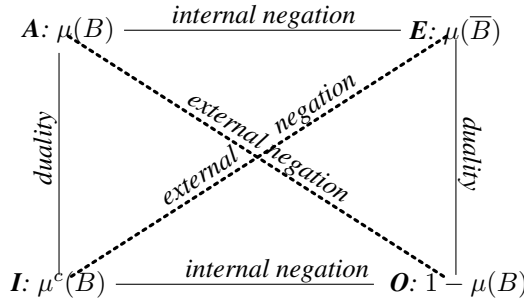


FIGURE 8. A modern square of opposition for capacities

We get a graded modern square in the general case, since the implication relation  $\mu(B) \leq \mu^c(B) = 1 - \mu(\bar{B})$  may not hold. We obtain a classical square if  $\mu(B) \leq \mu^c(B)$ . If  $\mu(B) \geq 1 - \mu(\bar{B})$ , it is a reversed graded classical square.

We end this subsection with the examples of (graded) modern squares induced by a qualitative integral defined from a capacity.

**Example 10.** We consider  $L$  a finite totally ordered scale equipped with an involutive order reversing map, denoted by  $n$  when used as an internal negation and by  $\neg$  when used as an external negation.  $\mathcal{C}$  is a finite set and  $\mu$  a capacity on the power set

<sup>2</sup>If we call  $\alpha, \epsilon, o, \iota$ , the numbers in  $[0, 1]$  that are the values of the expressions associated with vertices  $\mathbf{A}, \mathbf{E}, \mathbf{O}, \mathbf{I}$  respectively, then contradictions is ensured by the relations  $o = 1 - \alpha$  and  $\iota = 1 - \epsilon$ , contrariety by  $\alpha + \epsilon \leq 1$ , sub-contrariety by  $\iota + o \geq 1$ , and implications by  $\alpha \leq \iota$  and  $\epsilon \leq o$ . This extends the logical relations in the classical square of opposition if we choose the connectives of Łukasiewicz logic. Namely,  $o = \neg\alpha = 1 - \alpha$ ,  $\iota = \neg\epsilon = 1 - \epsilon$ ,  $\alpha \wedge \epsilon = \max(0, \alpha + \epsilon - 1) = 0$ ,  $\iota \vee o = \min(1, \iota + o) = 1$ ,  $\alpha \rightarrow \iota = \min(1, 1 - \alpha + \iota) = 1$  and  $\epsilon \rightarrow o = \min(1, 1 - \epsilon + o) = 1$  [5].

$2^{\mathcal{C}}$ . Let  $x : \mathcal{C} \rightarrow L$  be a mapping; typically  $x = (x_1, \dots, x_i, \dots, x_N)$  is an evaluation in the sense of a set of criteria  $\mathcal{C} = \{1, \dots, N\}$ . Define  $x_{\inf}, x_{\sup} : 2^{\mathcal{C}} \rightarrow L$  as follows:

$$x_{\inf}(T) = \min_{i \in T} x_i \text{ and } x_{\sup}(T) = \max_{i \in T} x_i.$$

Given a qualitative capacity  $\mu$  that provides a weighting system on the evaluation criteria accounting for their importance and mutual synergies, we consider the qualitative integral<sup>3</sup>

$$S(\mu, x) = \min_{T \in 2^{\mathcal{X}}} \max(\mu(\overline{T}), x_{\sup}(T))$$

It induces the square of opposition of Figure 9 [17]. Note that **I** is nothing but  $S(\mu^c, x)$  where  $\mu^c$  is the conjugate capacity<sup>4</sup>. This is clearly a modern square of opposition. This is not a classical square since, in general, there is no inequality between  $S(\mu, x)$  and  $S(\mu^c, x)$ .

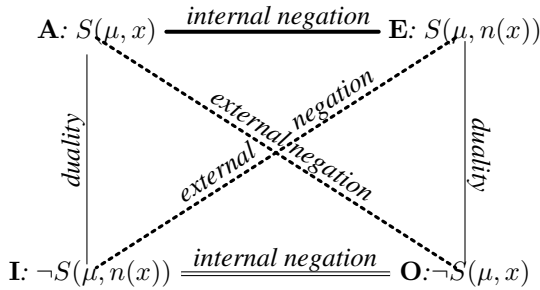


FIGURE 9. Modern square with qualitative integrals

We can obtain another modern square by applying negation  $n$ , the order reversing map of  $L$ , to the capacity  $\mu$ .  $n(\mu)$  is an anticapacity and this gives birth to so-called desintegrals [12, 17].

$$\begin{array}{ll} \mathbf{A}: S(\mu, x) & \mathbf{E}: S(n(\mu), x); \\ \mathbf{I}: \neg S(n(\mu), x) & \mathbf{O}: \neg S(\mu, x). \end{array}$$

In the next section, we investigate the idea of modern hexagons, and then show the interest of graded hexagons for quantitative expressions, in the spirit of Example 9 (and Figure 8).

### 3. Hexagons

In order to propose a modern counterpart to the classical hexagon of opposition, we first need to present a modern square based on statements  $S(A, B)$  involving two identified arguments. Then we shall provide a brief reminder on the classical hexagon, before introducing the modern one.

<sup>3</sup>This is one of the possible expressions of a  $L$ -valued Sugeno integral, which is an important family of aggregation functions [39, 21, 19].

<sup>4</sup>Indeed the integral can be equivalently written  $S(\mu, x) = \max_{T \in 2^{\mathcal{X}}} \min(\mu(T), x_{\inf}(T))$ .

**3.1. Preliminaries**

Now we consider that  $S$  depends on two arguments  $A$  and  $B$ . Then internal negations may take the forms  $S(n_1(A), B)$ ,  $S(A, n_2(B))$ , or  $S(n_1(A), n_2(B))$ , where  $n_1$  and  $n_2$  are involutive internal negations respectively for the first and the second argument. These internal negations lead to three possible modern squares of opposition. One is displayed in Fig. 10 (the internal negation is on the second predicate). Strictly speaking, the change from  $S(A, B)$  into  $\neg S(A, n_2(B))$  (or into  $\neg S(n_1(A), B)$ ) is a *semi-duality* (following the terminology used in [12]), even if shall continue to simply speak of duality.

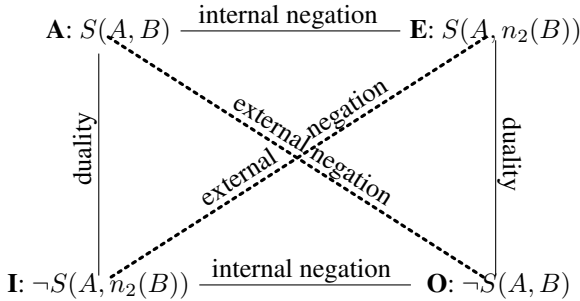


FIGURE 10. A modern square of opposition

Note that the case with internal negation of  $S(A, B)$  defined as  $S(n_1(A), n_2(B))$  with  $n_1 = n_2 = \neg$  corresponds to a Klein group of four logical transformations of a given logical statement  $\phi = f(p, q)$ : (i) the identity  $I(\phi) = \phi$ , (ii) the negation  $N(\phi) = \neg\phi$ , (iii) the reciprocation  $R(\phi) = f(\neg p, \neg q)$ , (iv) the correlation  $C(\phi) = \neg f(\neg p, \neg q)$ , identified by Piaget [31]; see also Gottschalk [20].

**Example 11.** *Blanché* [4] has provided an example of a classical square of opposition that exhibits inequalities between numbers, and which is at the basis of one of his most famous examples of hexagon. See Figure 11. Note that an internal negation of the form  $S(n_1(A), n_2(B))$  (where  $A$  and  $B$  are two numbers in the unit interval  $[0, 1]$  with  $n_1(C) = n_2(C) = 1 - C$ ) is at work in Figure 11. Indeed, since  $A < B \Leftrightarrow 1 - A > 1 - B$  and  $A \leq B \Leftrightarrow 1 - A \geq 1 - B$ ,  $A$  and  $B$  are thus exchanged when we go from left hand side vertices to right hand side vertices. Thus, this square is also a modern one.

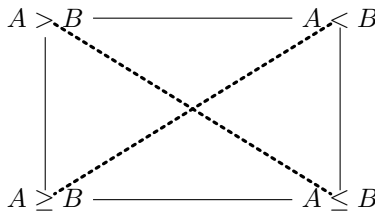


FIGURE 11. Modern square induced by two internal negations

Note that the square of opposition in Figure 11 remains a classical square for any real numbers  $A$  and  $B$  ( $[0, 1]$  is now replaced by the whole real line). Assuming that the internal negation is now taken as  $n(S(A, B)) = S(B, A)$ , it is also a modern square.

### 3.2. The classical hexagon of opposition

This subsection is a reminder about the hexagon of opposition proposed by Blanché [4]. This structure is anticipated in the works of three forerunners: Nelson<sup>5</sup> [29, 1], Jacoby [22], and Sesmat [36]. Blanché noticed that adding two vertices  $U$  and  $Y$  respectively defined as the disjunction of  $A$  and  $E$ , and the conjunction of  $I$  and  $O$ , respectively, one obtains a hexagon  $AUEOYI$  which contains 3 classical squares of opposition,  $AEOI$ ,  $YAUO$ ,  $YEUI$ .

Such a hexagon is obtained each time a tripartition of mutually exclusive situations such as  $A$ ,  $E$  and  $Y$  that play the same role is considered [9]. The tripartition underlying the hexagon is already mentioned by Sauriol [34]. Figure 12 shows the well-known example based on comparators of such a hexagon. Note that the square  $AEOI$  is the square of Figure 11.

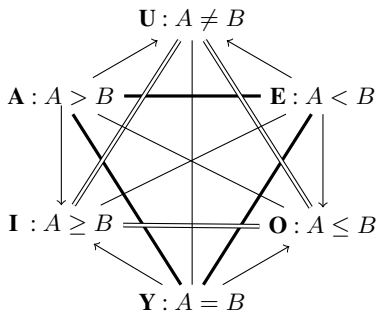


FIGURE 12. Hexagon induced by a complete pre-order.

In any classical hexagon, the following relations hold

- the diagonals linking two vertices express contradictions between the statements associated to these vertices;
- vertices  $A$ ,  $E$ ,  $Y$  are linked by contrariety links;
- vertices  $U$ ,  $O$ ,  $I$  are linked by sub-contrariety links;
- links between two successive vertices are implication links:  $A$  implies  $U$ ,  $E$  implies  $U$ ,  $Y$  implies  $O$ ,  $Y$  implies  $I$ ,  $A$  implies  $I$ ,  $E$  implies  $O$ .

This is exemplified by the example of Figure 12.

Another example of hexagon is presented in [22] in terms of hypothesis tests, where the semantics of the vertices are probabilistic thresholded expressions. By introducing two ordered thresholds, the authors are led to propose nested hexagons of opposition. The vertices of the outside hexagon and of the inside hexagon are related

<sup>5</sup>Nelson, a post Kantian philosopher, indeed in his 1921 lectures [29], seems to be the first to use hexagonal diagrams for discussing opposition between abstract notions, in some places. In his hexagons, the opposite vertices appear to be contradictory. In the introduction of his translation of Nelson lectures, Leal has offered an abstract rendering of the diagrams, which appear to be topologically equivalent to Blanché’s hexagon [2]. Yet, in Leal version of the diagram, it does not seem that the vertices corresponding to  $A$  and  $E$  should be contraries.

by implications. Such an idea of a nested structure can be implemented already in squares, for instance in Examples 5 or 7, with thresholded expressions.

**3.3. Modern hexagon**

The modern hexagon of opposition, which, as we shall see, still contains 3 (modern) squares, is not obtained from the modern square in the same way as the classical hexagon can be derived from the classical square. Namely, the additional vertices are not obtained as logical combinations of other vertices.

Rather, considering a two place relation  $S(A, B)$ , on the vertex  $\mathcal{U}^6$ , with two identified arguments  $A$  and  $B$ , we define a hexagon as follows. We generate two modern squares:

- one considering the internal negation  $n_1$  acting on the first argument  $A$ ,
- one considering the internal negation  $n_2$  acting on the second argument  $B$ .

These two modern squares  $\mathcal{U}\mathcal{O}\mathcal{Y}\mathcal{A}$  and  $\mathcal{U}\mathcal{I}\mathcal{Y}\mathcal{E}$  have two common vertices bearing  $S(A, B)$  and  $\neg S(A, B)$ . This is pictured in Figure 13 which exhibits a modern hexagon (this display is called type 1.A). As can be checked, while these two modern squares are respectively defined from a  $n_1$ -based internal negation and a  $n_2$ -based internal negation, there is a third (horizontal) square in the hexagon where both internal negations  $n_1, n_2$  are at work. This is a modern square based on the internal negation defined by the double application of  $n_1$  and  $n_2$ . Diagonals in the hexagon still link contradictory terms. The edges of the hexagon exhibit, alternatively, 3 forms of relations between expressions involving internal negations  $n_1$  alone, or  $n_2$  alone, or both  $n_1$  and  $n_2$  together with the external negation  $\neg$  (leading to a duality). The two triangles  $\mathcal{I}\mathcal{U}\mathcal{O}$  and  $\mathcal{A}\mathcal{Y}\mathcal{E}$  inside the hexagon respectively put together the positive vertices bearing  $S(A, B), S(n_1(A), B), S(A, n_2(B))$  on the one hand, and those bearing their negation on the other hand, as in Figure 13. Their six sides correspond to the applications of  $n_1$ -,  $n_2$ - and  $n_1, n_2$ -based internal negations respectively in each of the 3 modern squares  $\mathcal{A}\mathcal{U}\mathcal{O}\mathcal{Y}, \mathcal{U}\mathcal{E}\mathcal{Y}\mathcal{I}$  and  $\mathcal{A}\mathcal{E}\mathcal{O}\mathcal{I}$ .

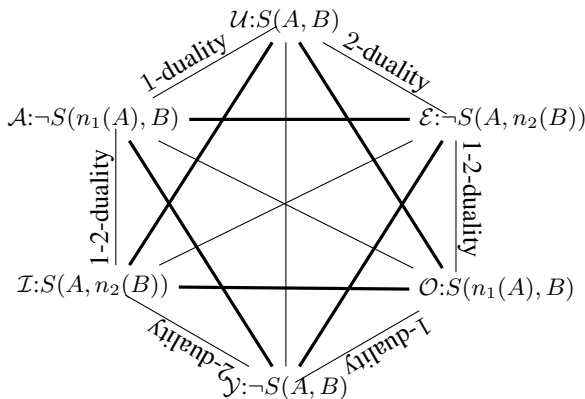


FIGURE 13. Modern hexagon - Type 1.A

<sup>6</sup>For modern hexagons, we continue to use the same names for the vertices, but we use calligraphic letters instead of standard capitals, since we are no longer expecting the same logical relations between the vertices as in the classical hexagon. In fact, as we shall see these letters are now just a matter of convenience for naming the vertices.

Along the perimeter of hexagon 1.A, the  $n_1, n_2$ -duality is between the  $n_2$ -duality and the  $n_1$ -duality. In the hexagon 1.B of Figure 14 a similar ordering can be observed exchanging  $n_1$  and  $n_2$ . Moreover the horizontal square is now based on  $n_2$ -duality. Hexagons 1.A and 1.B are clearly isomorphic.

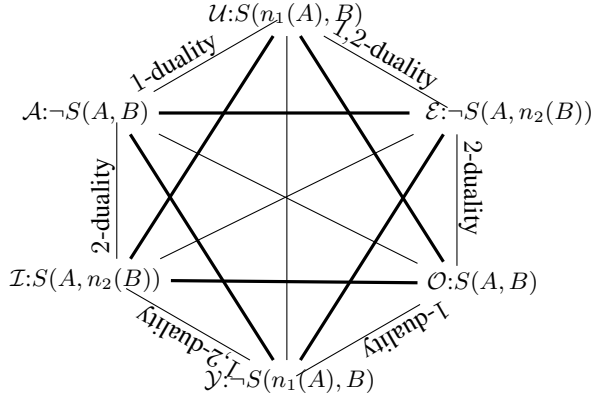


FIGURE 14. Modern hexagon - Type 1.B

There are clearly other isomorphic variants of hexagons 1.A and 1.B, preserving the alternation of positive and negative expressions, such as, e.g., hexagons with the square induced by  $n_1$  negations horizontal, but these hexagons can be obtained from 1.A or hexagon 1.B, by rotations and/or renaming of expressions (e.g. letting  $S'(A, B) = S(n_1(A), B)$ , and thus  $S'(n_1(A), B) = S(A, B)$  and so on).

In Figures 13 and 14, positive and negative vertices alternate. A slightly different display of the hexagon can be obtained by putting all the negative vertices on “the same side”, as in Figure 15 (type 2 modern hexagon). In this case, the  $n_1$ - and  $n_2$ -based transformations on the perimeter of the hexagon are simple internal negations and no longer involve the external negation. In Figure 15 note that  $n_1$ -duality and  $n_2$ -duality are at work along the non horizontal sides of the two triangles  $IYO$  and  $AUE$ , and the  $n_1, n_2$ -based negation along their horizontal sides.

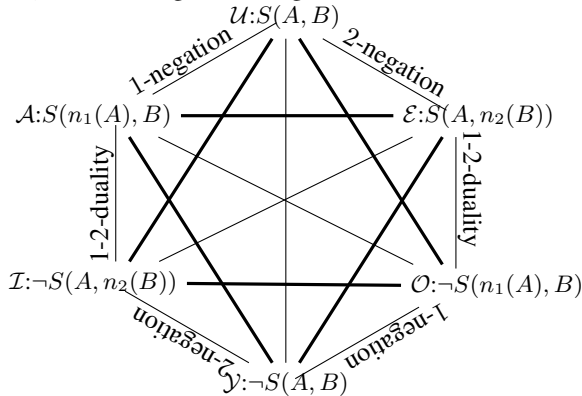


FIGURE 15. Modern hexagon - Type 2

In the same way as we move from type 1 hexagons to type 2 hexagons by putting all the negative vertices on “the same side”, we can get a fourth modern

hexagon from the hexagon of Figure 14 (type 1.B). The result is pictured in Figure 16. Here,  $n_1$ -negation,  $n_1, n_2$ -negation, and  $n_2$ -duality are along the perimeter, while  $n_2$ -negation,  $n_1$ -duality,  $n_1, n_2$ -duality are at work along the sides of the two inside triangles.

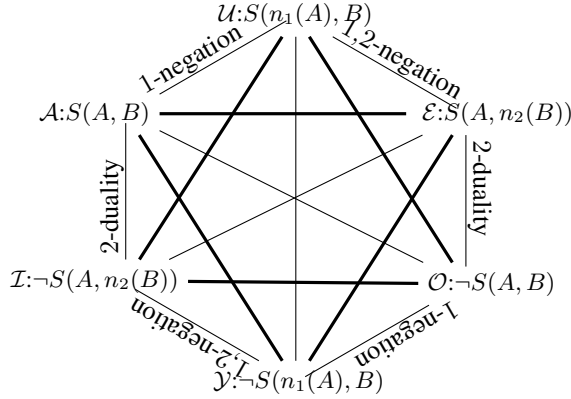


FIGURE 16. Modern hexagon - Type 3

Clearly, there are also hexagons where  $n_2$ -negation,  $n_1, n_2$ -negation,  $n_1$ -duality are at work along the perimeter (thus exchanging the roles of  $n_1$  and  $n_2$  with respect to the hexagon of Figure 16).

Note that it is impossible that the  $n_1$ -negation, the  $n_2$ -negation, and the  $n_1, n_2$ -negation be all at work along the perimeter of the hexagon, because we need to generate negative expressions. It is also impossible to have two dualities and one negation at work along the perimeter since their successive applications would make it impossible to have contradictory expressions at the extremities of diagonals. Thus we have reviewed all the possibilities, namely only the 3 dualities along the perimeters as in types 1.A and 1.B hexagons, or 1 duality and 2 negations (as in types 2 and 3 hexagons).

All these hexagons are isomorphic in the sense that they exhibit, two times, the effect of  $n_1$ -negation,  $n_2$ -negation,  $n_1, n_2$ -negation, and of  $n_1$ -duality,  $n_2$ -duality,  $n_1, n_2$ -duality, either along the 6 edges of the hexagon, or along the 3 sides of each of 2 triangles. In fact, speaking of hexagons here is just a matter of display. What we really have is a complete graph with 6 vertices where each vertex is connected to the 5 other ones through the edges and diagonals of the hexagon and the sides of the two inner triangles.

Moreover, it is worth noticing that, in such a graph, the combination of transformations along a path yields the same result as the direct transformation exchanging the two expressions at the extremities of the path. Indeed the composition of two dualities yields a negation (e.g. the composition of the  $n_1$ -duality with the  $n_2$ -duality yields the  $n_1, n_2$ -negation), the composition of a negation and a duality yields a duality (e.g., the  $n_1, n_2$ -negation with the  $n_2$ -duality gives the  $n_1$ -duality), the composition of  $n_1$ -duality,  $n_2$ -duality and  $n_1, n_2$ -duality or the composition of 1 duality with the 2 complementary negations yields the external negation (e.g., the composition of  $n_2$ -duality with the  $n_1$ -negation and the  $n_1, n_2$ -negation), the composition of the external negation with an internal negation yields a duality, and the composition



of the external negation with a duality gives an internal negation. In other words, the compositions involving the three internal negations, the external negation, plus the three dualities, and the identity transformation, seem to form an algebraic structure (a group of transformations) that captures the essence of the various displays of the modern hexagon of opposition. It can be viewed as an extension of Piaget group of transformations from 4 to 8 elements.

**Remark.** Pfeifer and Sanfilippo [30] have also proposed a probabilistic hexagon of opposition starting with statements  $S(A, B)$  of the form  $P(B|A) \geq \alpha$ . Their (classical) hexagon is obtained in the usual way by logical combination of vertices. For instance, vertex **U** is associated with the statement  $P(B|A) \geq \alpha \vee P(\overline{B}|A) \geq \alpha$ . It is clear that such a statement cannot be obtained in a modern hexagon of any type, where statements of the form  $P(B|\overline{A}) \geq \alpha$  would rather appear. Still, it would horizontally include the classical square of Figure 6 (for probabilities conditioned by  $A$ ), corresponding to the  $n_2$ -based duality acting on  $B$ .

In the following we rather consider quantitative expressions for providing an example of modern hexagon.

### 3.4. Graded modern hexagon

In [14, 15], we have considered a graded hexagon of opposition for ordered weighted min and max aggregation operations. Under some conditions, it turns to be a classical hexagon. As we shall see, without any additional condition, it is also a modern hexagon. Let us first recall this graded hexagon.

**Example 12.** *Ordered weighted min and max aggregation operations (abbreviated in  $OWmin$  and  $OWmax$ , see, e.g., [7]) are weighted versions of min and max operations that can express quantifiers such as at least  $k$ . For instance, the  $OWmin$  takes the minimum of the  $k$  largest values to be aggregated.*

*Let  $\{1, \dots, N\}$  be a set of attributes, and  $x \in [0, 1]^N$  be a vector of evaluations of some object according to these  $N$  attributes. Let  $\sigma$  be the permutation of attributes such that  $x_{\sigma(1)} \geq \dots \geq x_{\sigma(N)}$ . Define  $\mu_k : \{1, \dots, N\} \rightarrow [0, 1]$  by  $\mu_k(i) = 1$  if  $1 \leq i \leq k$  and 0 otherwise. Then, we have:*

$$OWmin_{\mu_k}(x) = \min_{i=1}^N \max(1 - \mu_k(i), x_{\sigma(i)})$$

and by De Morgan duality we define  $OWmax$

$$OWmax_{\mu_k}(x) = 1 - OWmin_{\mu_k}(1 - x) = \max_{i=1}^N \min(\mu_k(i), x_{\sigma(i)})$$

where  $\sigma$  is now such that  $x_{\sigma(1)} \leq \dots \leq x_{\sigma(N)}$ .

We can then build the following hexagon pictured in Figure 17. As we are going to see, this is both a graded hexagon<sup>7</sup> (provided that a condition holds, as explained below), and a modern square.

<sup>7</sup>The contradictory, contrariety, sub-contrariety, and implication relations between the graded expressions associated to the vertices of the hexagon are defined with the same multiple-valued logic connectives as the ones used for the graded square, recalled in footnote 2.

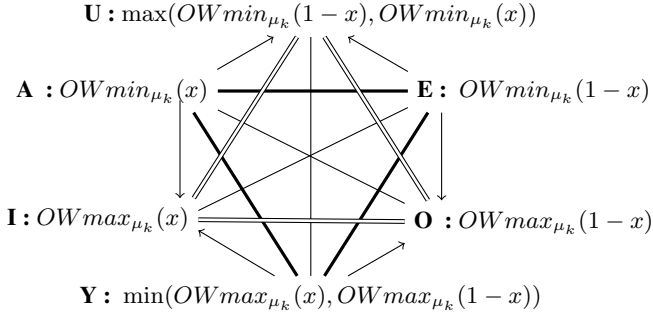


FIGURE 17. OWmin-OWmax hexagon

**Proposition 1.** *The hexagon AUEOYI of Figure 17, induced by the ordered weighted min operation over  $N$  numbers in  $[0, 1]$  with weights encoding the quantifier ‘at least  $k$ ’, is a graded hexagon of opposition provided that the condition  $2k \geq N + 1$  holds.*

*Proof.* First, De Morgan duality ensures that diagonals relate contradictories through the external negation  $1 - (\cdot)$ ; there are some straightforward implication links due to obvious inequalities: **A** implies **U**, **E** implies **U**, **Y** implies **O**, and **Y** implies **I**.

Unfortunately, **A** implies **I** and **E** implies **O** may not hold, since the inequality  $OWmin_{\mu_k}(x) \leq OWmax_{\mu_k}(x)$  does not always hold. Indeed, assume  $k$  is small, then  $OWmin_{\mu_k}(x)$  is the minimum of a small number of  $x_{\sigma(i)}$  among the largest ones, while  $OWmax_{\mu_k}(x)$  is the maximum of a small number of  $x_{\sigma(i)}$  among the smallest ones. So there is no guarantee that this maximum is larger than the previous minimum. In contrast, if  $k$  is large enough, the minimum and the maximum are taken over most of the  $x_{\sigma(i)}$ ’s and the above inequality may hold. More precisely, as can be checked, the inequality  $OWmin_{\mu_k} \leq OWmax_{\mu_k}$  holds only when  $2k \geq N + 1$ , and then it holds that **A** implies **I** and **E** implies **O**. When  $2k \leq N + 1$ , implications are reversed: **I** implies **A** and **O** implies **E**. In other words, **AEOI** is a reversible square. This may be viewed as a graded counterpart of Westerståhl’s square in Figure 7.

Under condition  $2k \geq N + 1$ , vertices **A**, **E**, **Y** are related by contrariety links: indeed  $OWmin_{\mu_k}(x) + OWmin_{\mu_k}(1-x) (= OWmin_{\mu_k}(x) + 1 - OWmax_{\mu_k}(x))$  is less than 1 if  $2k \geq N + 1$ . For the link between **A** and **Y**, we have  $OWmin_{\mu_k}(x) + \min(OWmax_{\mu_k}(x), OWmax_{\mu_k}(1-x)) \leq OWmin_{\mu_k}(x) + OWmax_{\mu_k}(1-x) = OWmin_{\mu_k}(x) + 1 - OWmin_{\mu_k}(x) = 1$ . For the link between **E** and **Y** the computation is similar,

Under condition  $2k \geq N + 1$ , vertices **I**, **U**, **O** are related by sub-contrariety links:  $OWmax_{\mu_k}(x) + OWmax_{\mu_k}(1-x) = OWmax_{\mu_k}(x) + 1 - OWmin_{\mu_k}(x) \geq 1$ . For the link between **I** and **U**, we have  $\max(OWmin_{\mu_k}(1-x), OWmin_{\mu_k}(x)) + OWmax_{\mu_k}(x) \geq OWmin_{\mu_k}(1-x) + OWmax_{\mu_k}(x) = 1$ . For the link between **U** and **O** the computation is similar.

Thus **UEYI** and **AUOY** are graded squares of opposition, while **AEOI** is a reversible square. The graded square **AEOI** and the hexagon of opposition **AUEOYI** is obtained when condition  $2k \geq N + 1$  holds  $\square$

**Proposition 2.** *The hexagon **AUEOYI** in Figure 17, induced by the ordered weighted min operation over  $N$  numbers in  $[0, 1]$  with weights encoding the quantifier ‘at least  $k$ ’, is a modern hexagon of opposition of type 2 (in the sense of Figure 15).*

*Proof.* Let the expression at vertex **U** of the OWmin-OWmax hexagon be read as  $S(A, B)$ , where  $S(\cdot, \cdot) = \max(\text{OWmin}_{\mu_k}(\cdot), \text{OWmin}_{\mu_k}(\cdot))$  and  $A = 1 - x, B = x$ , letting  $n_1(x) = n_2(x) = 1 - x, \neg S = 1 - S$ . To check we do have a type 2 modern hexagon, we need to check the other vertices:

- Vertex **A**:  $S(n_1(A), B) = \text{OWmin}_{\mu_k}(x)$  (thanks to the idempotency of min).
- Vertex **I**:  $\neg S(A, n_2(B)) = 1 - \max(\text{OWmin}_{\mu_k}(1 - x), \text{OWmin}_{\mu_k}(1 - x)) = \text{OWmax}_{\mu_k}(x)$ .
- Vertex **Y**:  $\neg S(A, B) = 1 - \max(\text{OWmin}_{\mu_k}(1 - x), \text{OWmin}_{\mu_k}(x)) = \min(\text{OWmax}_{\mu_k}(x), \text{OWmax}_{\mu_k}(1 - x))$ .
- Vertices **E, O**: Same as vertices **A, I**, replacing  $n_1$  by  $n_2$ .

Thus, the hexagon **AUEOYI** of Figure 17 is also a modern hexagon of type 2 in the sense of Figure 15.  $\square$

A similar construction as in the previous Example 17 enables us to build a hexagon for capacities from the square of Figure 8 in Example 9.

**Example 13.** *The square **AEOI** in Figure 8 is a classical square of opposition as soon as  $\forall A, \mu(A) \leq \mu^c(A)$ . Such an inequality defines the so-called pessimistic capacities [10]. In case the inequality is reversed, i.e.,  $\forall A, \mu(A) \geq \mu^c(A)$ , the capacity  $\mu$  is said to be optimistic and the square of opposition is reversed. Thus, the hexagon **AUEOYI** is a gradual hexagon of opposition if  $\mu$  is a pessimistic capacity. Indeed **U** and **Y** relate contradictories since the associated expressions are exchanged through the application of  $1 - (\cdot)$  taken as an external negation; it is easy to check that  $\mu(B) + \min(\mu^c(B), \mu(\overline{B})) \leq 1$  and so on. The hexagon **AUEOYI** is also a modern square of opposition (of type 2) whatever the capacity  $\mu$ . The proof is similar to the one of Proposition 2, writing the expression associated with **U** as  $S(A, B)$  with  $S(\cdot, \cdot) = \max(\mu(\cdot), \mu(\cdot))$ , where  $n_1 = n_2$  is the set complement, and  $\neg$  is  $1 - \cdot$ . Then **A** and **I** as well as **E** and **O** are exchanged by  $n_1, n_2$  duality, i.e., applying the internal negations  $n_1$  and  $n_2$  and taking the (external) negation of the result, as in the case of hexagon of Figure 17.*

Another, modern, hexagon can be built from the square of opposition for capacities in Figure 8 of Example 9.

**Example 14.** *Not all capacities are pessimistic (or optimistic). But, given a capacity  $\mu$ , one can consider its pessimistic part  $\min(\mu, \mu^c)$  and its optimistic part  $\max(\mu, \mu^c)$  [10]. This gives birth to the hexagon **AUEOYI** of Figure 19 where diagonals link contradictories in the sense of  $1 - (\cdot)$ . Note that we have inverted the respective place of  $\mu^c(B)$  and  $\mu(\overline{B})$  in the horizontal square with respect to Figure*

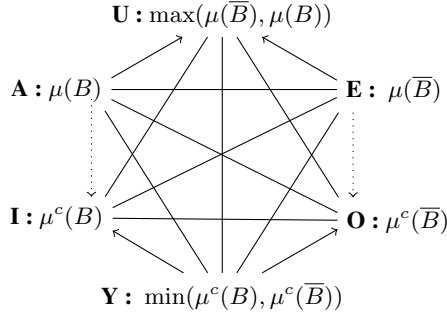


FIGURE 18. A possibly classical hexagon and a type 2 modern hexagon for capacities

8. It is not a classical square since there cannot exist any general inequality between  $\mu(B)$  and  $\mu(\bar{B})$ .

$AU\mathcal{E}OY\mathcal{I}$  is however a modern hexagon, as we are going to see. Before explaining it, a warning is necessary. In all the examples of this paper the internal negations used are “true negations” (i.e., based on set complements, or on  $1 - (\cdot)$ ). In the current example it would not be the case. Indeed, strictly speaking, in a modern square, in a modern hexagon, the only requirement for an “internal negation” is to be involutive. The transformation  $n(\mu) = \mu^c$  is involutive. In the following, we take  $n_1(\mu) = n_2(\mu) = \mu^c$ . We shall speak of  $n_1$ -involution and  $n_2$ -involution (rather than of  $n_1$ - and  $n_2$ -negations) since mathematically speaking,  $\mu$  and  $\mu^c$  are dual of each other.

Consider the expression at vertex  $\mathcal{U}$ . It is of the form  $S(\cdot, \cdot) = \max(\mu(\cdot), \mu^c(\cdot))$ . On Figure 19, one may qualify the edges of the hexagon as follows,

- from  $\mathcal{U}$  to  $\mathcal{A}$ : it is an  $n_2$ -involution :  $\max(\mu(B), \mu^c(B))$  is changed into  $\mu(B)$  (changing  $\mu^c$  into  $\mu$ );
- from  $\mathcal{A}$  to  $\mathcal{I}$ : it is an  $n_1, n_2$ -duality. Indeed  $\max(\mu(B), \mu(B))$  is changed into  $1 - \max(\mu^c(B), \mu^c(B)) = \min(1 - \mu^c(B), 1 - \mu^c(B)) = \min(\mu(\bar{B}), \mu(\bar{B})) = \mu(\bar{B})$ ;
- from  $\mathcal{I}$  to  $\mathcal{Y}$ : it is an  $n_1$ -involution;
- from  $\mathcal{U}$  to  $\mathcal{E}$ : it is an  $n_1$ -involution;
- from  $\mathcal{E}$  to  $\mathcal{O}$ : it is an  $n_1, n_2$ -duality (same as from  $\mathcal{A}$  to  $\mathcal{I}$ );
- from  $\mathcal{O}$  to  $\mathcal{Y}$ : it is an  $n_2$ -involution.

So this is a type 2 modern hexagon.

Other modern hexagons could be built starting from modern squares in a way similar to hexagons 17, 18 and 19. The modern structure is quite versatile, as shown by the case of capacities where two modern hexagons are built, while one classical hexagon has been obtained under specific conditions relating the capacity and its conjugate. Also notice the key-role of idempotency of the connectives that create formal expressions on vertices  $\mathcal{U}, \mathcal{Y}$  from those on the horizontal square vertices:

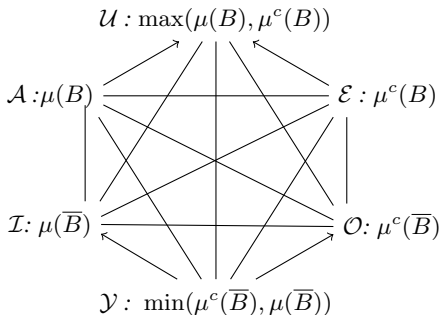


FIGURE 19. Another type 2 modern hexagon for capacities

changing min and max into product and its dual  $a + b - ab$  in Figures 18 and 19, for instance, would not work.

### 4. Modern cubes of opposition

Let us briefly examine modern counterparts of cubes of opposition. As can be seen in the previous section, the 6 vertices of the hexagons do not exhaust all the possible expressions induced from  $S(A, B)$  by application of the two internal negations and the external one. For instance, expressions of the form  $S(n_1(A), n_2(B))$  and  $\neg S(n_1(A), n_2(B))$  are missing from Types 1, 2, 3 displays. Integrating the 8 possible expressions in a single structure requires a cube! In this section we investigate this type of structure putting it in a “modern” perspective. We first recall the two main types of cube of opposition which have emerged in the literature.

#### 4.1. Existing cubes

Moretti [25, 26] has proposed a cube of opposition<sup>8</sup> that is arguably the extension to a quadri-partition of the situation encountered with the classical hexagon induced by a tripartition. An example of this cube is exhibited in Figure 20 with the 4 mutually exclusive comparison relations:  $A$  is preferred to  $B$  ( $A > B$ ),  $B$  is preferred to  $A$  ( $A < B$ ),  $A$  is equivalent to  $B$  ( $A = B$ ),  $A$  and  $B$  are not comparable ( $A \succ\prec B$ ) and their respective negations:  $A \not> B$ ,  $A \not< B$ ,  $A \neq B$ ,  $A \not\sucprec B$ . In this cube, diagonals relate contradictories, and all edges are implications. Thick lines mean mutual exclusiveness, double lines mean ‘may be true together’.

Another type of cube of opposition, called JK-cube<sup>9</sup>, is obtained by associating two classical squares of opposition, which are the front and back facets of the cube. The cube involves expressions with two arguments, say  $A$  and  $B$ . The front facet involves negation of the relation linking  $A$  and  $B$  and / or a negation applied to  $B$ ; the back facet contains similar expressions, except that they now all involve

<sup>8</sup>However, an isomorphic cube already appeared in the middle of last century in a discussion of syllogisms [33], see [16] for details.

<sup>9</sup>What we call here “JK cube” was in fact presented as an octagon by Johnson [23] and Keynes [24]. It has been rediscovered several times, e.g., [9], [40].

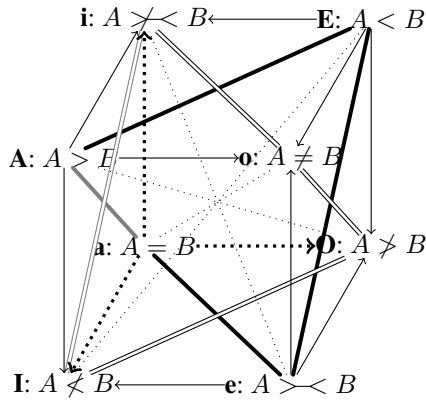


FIGURE 20. Moretti's cube of opposition

the negation of  $A$ . This is exemplified in Figure 21 where the cube is induced by set inclusion ( $A, B$  and their complements are assumed to be non empty sets). Note that the square of the back facet is upside down, that the edges of the side facets are implications, and that contradictories are related by the diagonals in the front and back facets.<sup>10</sup>

The JK cube is different from Moretti's cube. The latter displays 12 implications (on all the edges of the cube), instead of only 8 for the JK cube (on the edges of its side facets). The reader is referred to [16] for more details on the logical characterization of each of the two cubes (and other related cubes), their comparison, and their gradual versions.

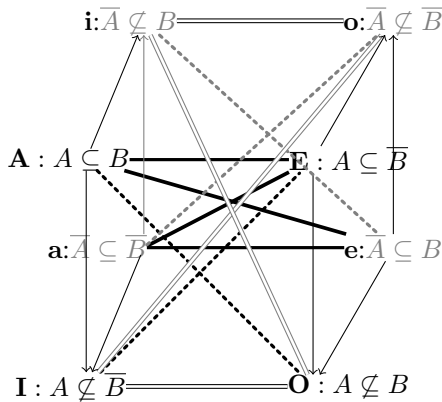


FIGURE 21. JK cube of opposition induced by inclusion

<sup>10</sup>This latter point is a matter of convenience and readability, and not a distinctive feature with respect to Moretti's cube. Indeed we may choose to use the front and back facets of the JK cube as the diagonal plans of a cube, just leading to a new isomorphic cube the diagonals of which relate contradictories.

### 4.2. Modern cube

Is there a “modern” reading of the two previous cubes? Let us first consider Moretti’s cube. This cube is characterized by 4 mutually exclusive expressions that lead to implications on all edges, while contradictories are linked by diagonals. Following the examples of moving from the classical hexagon to the modern one, we have to interpret the edges as supporting dualities or internal negation transformations, while the diagonals continue to link contradictories. Let us see an example.

**Example 15.** *We consider an example from [16], displayed on Figure 22. Vertices are associated with relative cardinalities that involve two subsets  $A$  and  $B$  of a universe  $U$ . It is a gradual cube of opposition. Indeed diagonals relate relative cardinalities of two subsets that complementary, so the relative cardinality of one is equal to 1 minus the complementary of the other. Since cardinality is a set increasing function, each implication corresponds to an inequality as can be checked.*

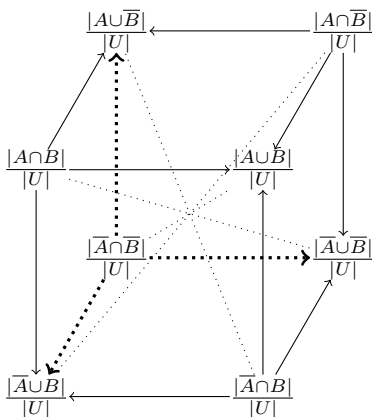


FIGURE 22. Moretti’s cube for relative cardinalities

The cube on Figure 22 and Example 15 may have a “modern” reading. Each expression is of the form  $S(A \perp B) = k|A^* \perp B^*|$  with  $X^*$  equal to  $X$  or to  $\bar{X}$ ,  $\perp$  standing for  $\cap$  or  $\cup$ ,  $k = 1/|U|$ . Let us denote by  $n_1$  (resp.  $n_2$ ) the set complementations on the first (resp. second) argument of  $A^* \perp B^*$ , and  $\neg$  the set complementation applying to the whole expression  $A^* \perp B^*$ , which plays the role of the external negation. Then the cube on Figure 22 can be represented by the cube on Figure 23.

In Figure 23, front and back facets exhibit  $n_1, n_2$  duality and  $n_2$  duality, top and bottom facets  $n_1, n_2$  duality and  $n_1$  duality, side facets  $n_1$  duality and  $n_2$  duality.  $n_1$  negation and  $n_2$  negation are “hidden” in the squares of diagonal plans.

Let us now consider JK-cubes. The cube on Figure 24 is the modern reading of the cube on Figure 21 (with the same notation conventions as in the previous cube, and where the general relation  $S$  replaces set inclusion). We can see that the cube of opposition induced by set inclusion of Figure 21 is thus both a classical and a “modern” cube.

In case we introduce a third internal negation, it would be possible to obtain a modern double cube, or tesseract; see [17].

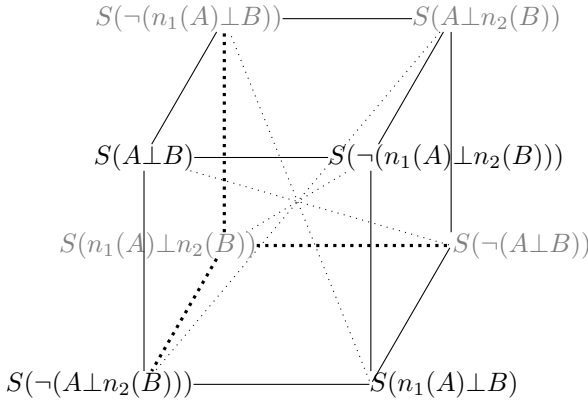


FIGURE 23. Modern counterpart of Moretti’s cube in Figure 22

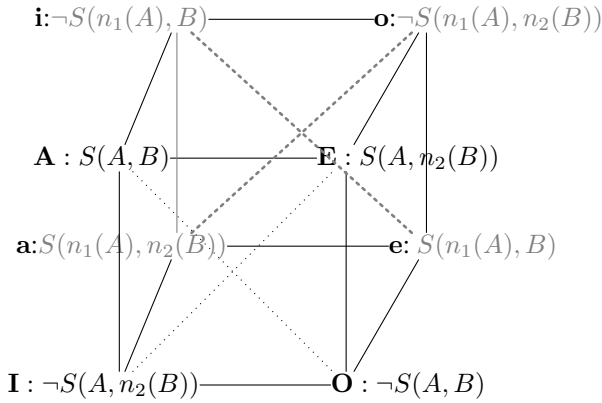


FIGURE 24. Modern counterpart of the JK cube of opposition

We can see in Figure 24 that the  $n_2$  negation and the  $n_2$  duality are at work in the front and back facets, while the  $n_1$  negation and the  $n_2$  negation are at work in the top and bottom facets, while  $n_1$  duality and the  $n_2$  duality are at work in the side facets.

We observe that the modern cubes obtained from Moretti’s cube and JK-cube are quite similar. In fact they are isomorphic, taking the front and back facets of the cube on Figure 24 as vertical diagonals of a cube. In other words, the modern reading seems to erase the differences between the two cubes. This is not a surprise, since this reading forgets logical relations such as implications.

We have provided two examples of “classical” cubes that also have a “modern” reading. We end this section with an example of “modern” cube without “classical” reading.

**Example 16.** *This example is about Sugeno integrals which are qualitative integrals already considered in the Example 10 of modern squares. We use the notations of*



this example, i.e., here  $S(A, B) = S(\mu, x)$ . We can build a cube with a JK display, where the front facet is the modern, non classical, square of Figure 9 (where  $\mu^c(T) = 1 - \mu(\bar{T})$ ). The back facet is the front facet upside down where the capacity  $\mu$  has been replaced by the anticapacity [12, 17]  $1 - \mu$ . We thus obtain 8 quantities in Figure 25, where the ones of the back facet are unfortunately equal to 0 (**i** and **o**) or 1 (**a** and **e**), as can be checked. So this modern cube is artificial, since it relies on syntactic transformations only. However it is possible to obtain a non trivial cube for Sugeno integrals  $S(\mu, x)$  where  $\mu$  is a necessity measure (see Figure 11 in [17]).

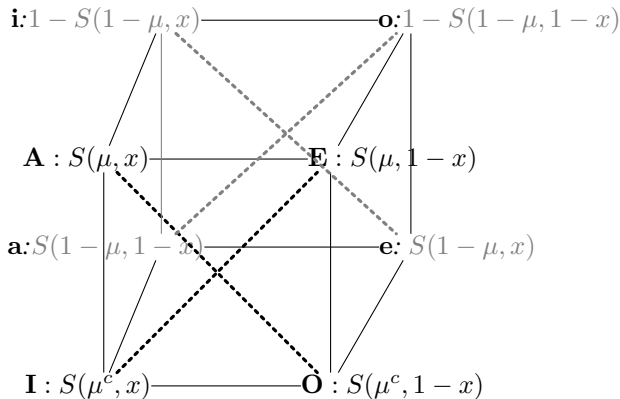


FIGURE 25. A modern cube for qualitative integrals

In case we introduce a third internal negation, it would be possible to obtain a modern double cube, or tesseract; see [17].

### 5. Concluding remarks

In this paper, we have proposed a formalization of “modern” squares, following an idea first proposed by Westersthåll, and we have extended this view to hexagons and cubes. We have investigated the differences between classical and “modern” structures of opposition. Several general lessons can be drawn from this study. Classical structures focus on expressions that have a truth value (or a degree of truth in the graded case) and the logical relations that hold true between these expressions. “Modern” structures do not consider the relationships between these expressions from such a perspective. They only examine transformations (involving internal negation(s) and / or an external negation) at work when we go from an expression to another. This latter point of view makes no difference between  $\{0, 1\}$ -valued expressions or graded expressions.

Besides, in the numerous examples of this paper, we have dealt with a variety of types of expressions: true or false statements such as relational statements of the form  $S(A, B)$  (between subsets or numbers), or quantitative expressions such as  $\mu(B) \geq \alpha$  involving a threshold, as well as graded expressions such as  $\mu(B)$

or such as  $\frac{|A \perp B|}{|U|}$  (where  $\perp$  is a set operation). In each case, we have shown the relevance of a modern reading.

We also have shown that classical squares of opposition may be reversed in the case of threshold expressions (they may not hold for some value of the threshold, while all cases are covered by the “modern” reading). Indeed in the “modern” view, the “subaltern” (implication) relation is replaced by a weaker duality relation, the mutual exclusiveness (contrariety) relation and the sub-contrariety (negative mutual exclusiveness) are weakened to a unique relation based on an internal (involutive) negation. This explains why “modern” structures agree with their classical counterparts in some cases, but not always.

Lastly, a bitstring encoding [6, 35] of logical formulas, or of relations, or still of subsets of criteria, has been shown successful for encoding structures of opposition. It is a convenient way for keeping track of the logical relations between vertices. As we have explained, the “modern” view, at least in the way we have developed its formalization, insists on the group of *transformations*, à la Piaget, necessary for moving from one vertex to another. How modern squares can also benefit from the bitstring encoding, is an open question.

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Didier Dubois

Institut de Recherche en Informatique de Toulouse (IRIT)  
Université Paul Sabatier, 118 route de Narbonne  
31062 Toulouse Cedex 9, France  
e-mail: [dubois@irit.fr](mailto:dubois@irit.fr)

Henri Prade

Institut de Recherche en Informatique de Toulouse (IRIT)  
Université Paul Sabatier, 118 route de Narbonne  
31062 Toulouse Cedex 9, France  
e-mail: [prade@irit.fr](mailto:prade@irit.fr)

Agnès Rico

Equipe de Recherche en Ingénierie des Connaissances (ERIC)  
Université Claude Bernard Lyon 1, Lyon, France  
e-mail: [agnes.rico@univ-lyon1.fr](mailto:agnes.rico@univ-lyon1.fr)