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# NOTES

Edited by Jimmie D. Lawson and William Adkins

## Pythagoras' Theorem for Areas

Jean-P. Quadrat, Jean B. Lasserre, and Jean-B. Hiriart-Urruty

The theorem of Pythagoras relating the squares of the lengths of the sides of a right triangle is one of the key results in elementary geometry. At a more advanced level, one learns that the Pythagorean theorem extends to prehilbert spaces, but it still expresses a relation among *lengths* of vectors.

We incidentally came across a result relating the areas of the faces of a right tetrahedron whose analogy with the Pythagorean theorem is striking. Here it is.

**Theorem 1.** *Let  $OABC$  be a tetrahedron with three perpendicular triangular faces  $OAB$ ,  $OAC$ ,  $OBC$ , and “hypotenuse-face”  $ABC$ ; see Figure 1. Let  $S_1$ ,  $S_2$ ,  $S_3$  denote the areas of the perpendicular faces and let  $S$  denote the area of the hypotenuse-face. Then*

$$S^2 = S_1^2 + S_2^2 + S_3^2. \quad (1)$$

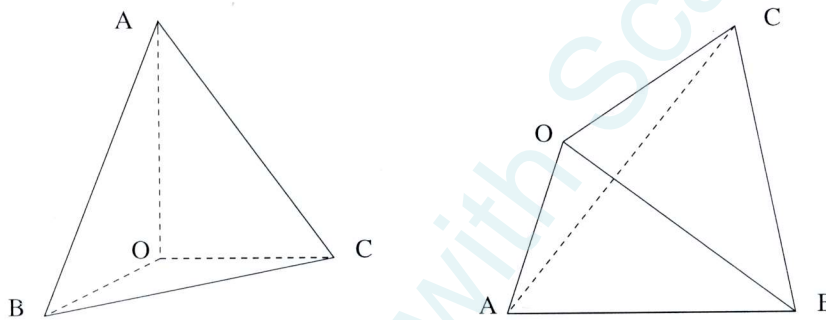


Figure 1. The right tetrahedron  $OABC$

Theorem 1 is very easy to prove; we derive a generalized version in the  $n$ -dimensional context.

When we came across the result of Theorem 1, which we propose to call *the Pythagorean theorem for areas*, we found it noteworthy and asked colleagues: is it original (of course we thought no ...)? is it well-known (maybe ...)? does it bear a name (who knows ...)? After several unsuccessful attempts, we finally found mention of this result; here is a brief historical account.

The result of Theorem 1 indeed appears in some old books of geometry, in chapters devoted to “Applications of vector calculus to analytical geometry in 3-dimensional space”; the areas are there calculated via vector products. A sample reference is [8, pp. 121–123]. According to [4, p. 98], it was very likely known to R. Descartes. Theorem 1 also appears in [7] (a “bible” for results in elementary geometry) where (p. 911–912) it is attributed to J.-P. Gua de Malves (published in his memoirs of 1783). However, the analogy with the Pythagorean theorem for lengths is nowhere pointed

out. Later on, in his famous work [3], L.N.M. Carnot states the result of Theorem 1 and refers to it as a “known result” [3, section 263, p. 311]; he also provides an extension to any convex polyhedron ([3, section 268, p. 313]), which is a generalization of the cosine theorem for triangles.

We now propose a generalization of Theorem 1 to  $n$ -dimensional spaces. Let the standard Euclidean affine space  $\mathbb{R}^n$  be marked off with  $(O; \vec{e}_1, \vec{e}_2, \dots, \vec{e}_n)$ , where  $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$  is an orthonormal basis of the vector space  $\mathbb{R}^n$ . We consider there the compact convex polyhedron (or polytope, or  $n$ -simplex)  $\Omega_n$  described as follows:

$$\Omega_n := \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n \frac{x_i}{a_i} \leq 1, x_i \geq 0, \text{ for all } i = 1, \dots, n \right\}, \quad (2)$$

where  $a_i > 0$  for all  $i = 1, \dots, n$ .

The “multi-orthogonal”  $\Omega_n$  is just a generalized version of the right tetrahedron of Theorem 1, and its structure from the convexity viewpoint is well-known ([1, p. 84], [2, p. 77–79], [6, p. 53]). Indeed,  $\Omega_n$  has

- $n + 1$  vertices: the origin  $O$  and the  $n$  points  $A_i$  defined as  $\vec{OA}_i = a_i \vec{e}_i$  for all  $i = 1, 2, \dots, n$ .
- $n + 1$   $(n - 1)$ -dimensional faces (also called *facets*):  $n$  of these facets contain the origin (obtained as convex hulls of  $O$  and  $n - 1$  points among the  $A_i$ 's); we call them *facets issuing from the origin*. One facet does not contain the origin (obtained as the convex hull of the  $A_i$ 's); we call it *hypotenuse facet*.

The  $(n - 1)$ -dimensional content of the facets of  $\Omega_n$  is called *area*, in analogy with the usual case with  $n = 3$ . The  $n$ -dimensional content of  $\Omega_n$ , called the “volume”  $V$  of  $\Omega_n$ , can be computed from the areas of facets by the following formula [1, p. 87]:

$$V = \frac{1}{n} (\text{area of a facet}) \times (\text{height from the vertex outside the considered facet}). \quad (3)$$

**Theorem 2. (an  $n$ -Dimensional Version of the Pythagorean Theorem for Areas)**

*For the compact convex polyhedron  $\Omega_n$  in (2), the square of the area of the hypotenuse facet is equal to the sum of the squares of the areas of the  $n$  facets issuing from the origin.*

For example, with  $n = 4$ , Theorem 2 gives a relation among volumes (in the usual sense of the word) of the four 3-dimensional faces:  $V^2 = V_1^2 + V_2^2 + V_3^2 + V_4^2$ .

*Proof.* To follow the proof, the reader is invited to take  $n = 3$  and keep in mind Figure 1. According to (3), if  $S_i$  denotes the area of the facet issuing from the origin “opposed” to the vertex  $A_i$ ,

$$V = \frac{1}{n} S_i \times \|\vec{OA}_i\| = \frac{1}{n} S_i a_i. \quad (4)$$

The height from the origin  $O$  to the hypotenuse facet is the distance from  $O$  to the affine hyperplane whose equation is  $\sum_{i=1}^n x_i/a_i = 1$  (the one containing all the vertices  $A_1, A_2, \dots, A_n$ ); it equals  $(\sum_{i=1}^n a_i^{-2})^{-1/2}$ . Hence, if  $S$  denotes the area of the

hypotenuse facet of  $\Omega_n$ , we infer from (3) that

$$V = \frac{1}{n} S \times \left( \sum_{i=1}^n \frac{1}{a_i^2} \right)^{-1/2}. \quad (5)$$

We infer from (4)

$$S_i^2 = n^2 V^2 \frac{1}{a_i^2} \quad \text{for all } i = 1, \dots, n$$

and from (5)

$$S^2 = n^2 V^2 \left( \sum_{i=1}^n \frac{1}{a_i^2} \right). \quad \blacksquare$$

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#### REFERENCES

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1. M. Berger, *Géométrie*, vol. 2, Nathan, Paris, 1990.
2. A. Brøndsted, *An Introduction to Convex Polytopes*, Springer-Verlag, New York, 1983.
3. L.N.M. Carnot, *Geométrie de Position*, Imprimerie Crapelet, Paris, 1803.
4. P. Costabel, *Descartes. Exercices pour la Géométrie des Solides (De Solidorum Elementis)*, Presses Universitaires de France, Paris, 1987.
5. P.S. Donchian and H.S.M. Coxeter, An  $n$ -dimensional extension of Pythagoras' Theorem, *Math. Gazette* **19** (1935) 206.
6. B. Grünbaum, *Convex Polytopes*, J. Wiley & Sons, London, 1967.
7. F. G-M, *Exercices de Géométrie, comprenant l'exposé des méthodes géométriques et 2000 questions résolues*, 6ième édition, J. Gabay, Paris, 1920, réédité par les Editions Jacques Gabay en 1991.
8. P. Nillus, *Leçons de calcul vectoriel. Tome I: Règles de calcul vectoriel - Applications à la géométrie élémentaire, à la trigonométrie et à la géométrie analytique*, Editions Eyrolles, Paris, 1931.

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