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# Bel-Games: A Formal Theory of Games of Incomplete Information Based on Belief Functions in the Coq Proof Assistant

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1 **Bel-Games:**  
2 **A Formal Theory of Games**  
3 **of Incomplete Information**  
4 **Based on Belief Functions**  
5 **in the Coq Proof Assistant\***

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12 — **Abstract** —

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13 Decision theory and game theory are both interdisciplinary domains that focus on modelling  
14 and analyzing decision-making processes. On the one hand, decision theory aims to account  
15 for the possible behaviors of an agent with respect to an uncertain situation. It thus provides  
16 several frameworks to describe the decision-making processes in this context, including that of  
17 belief functions. On the other hand, game theory focuses on multi-agent decisions, typically with  
18 probabilistic uncertainty (if any), hence the so-called class of Bayesian games. In this paper, we  
19 use the Coq/SSReflect proof assistant to formally prove the results we obtained in [29]. First, we  
20 formalize a general theory of belief functions with finite support, and structures and solutions  
21 concepts from game theory. On top of that, we extend Bayesian games to the theory of belief  
22 functions, so that we obtain a more expressive class of games we refer to as Bel games; it makes it  
23 possible to better capture human behaviors with respect to lack of information. Next, we provide  
24 three different proofs of an extended version of the so-called Howson–Rosenthal’s theorem, showing  
25 that Bel games can be turned into games of complete information, i.e., without any uncertainty.  
26 Doing so, we embed this class of games into classical game theory and thus enable the use of existing  
27 algorithms.

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29 computation → Type theory; Theory of computation → Higher order logic; Theory of computation  
30 → Algorithmic game theory; Theory of computation → Solution concepts in game theory; Theory  
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## 1 Introduction

From a mathematical perspective, measure theory is a fundamental domain to learn and use, notably given its direct application to integration and probability theory. Several works thus focused on formalizing measure theory in type theory, e.g., relying on reference textbooks [14]. Next, probability play a key role in the context of game theory, gathering several multi-agent frameworks that can model situations in many application areas such as economics, politics, logics, artificial intelligence, biology, and so on. In particular, the framework of Bayesian games (a class of games of incomplete information), has been well-studied by the decision theory community [13, 25]. However, using probability and additive measures appears to be unsatisfactory to model subtle decision-making situations with uncertainty.

In this work, we aim to show that the belief function theory also is amenable to formal proof, and makes it possible to formally verify the correctness of three state-of-the-art algorithms. In [11, 29], we introduced the notion of *Bel games*, which faithfully models games of incomplete information where the uncertainty is expressed within the *Dempster-Shafer theory of belief functions*. This framework naturally encompasses Bayesian games, as belief functions generalize probability measures. Also, we generalized the *Howson-Rosenthal theorem* to the framework of Bel games and proposed three transforms which make it possible to cast any Bel game into an equivalent game of complete information (without any uncertainty). Furthermore, these transforms preserve the space complexity of the original Bel game (they produce a game with a succinct representation, corresponding to the class of so-called hypergraphical games).

**Contributions.** In the present paper, we consolidate the mathematical results previously published in [29], presenting a formal verification of our algorithms using the Coq proof assistant [6]. First, we formalize a general theory of belief functions. Then, we formalize structures and solution concepts for “standard” games, Bayesian games, and Bel games, and we formally prove the correctness of the three transform algorithms, in order to provide strong confidence on these results. The software artifact obtained was released under the MIT license and is available within the official Coq projects OPAM archive. Our formalization effort also resulted in more background lemmas, integrated in the **MathComp** library. To the best of our knowledge, it is the first time the theory of belief functions is mechanized in a formal proof assistant, and applied to the domain of (formal) game theory of incomplete information.

**Related works.** Several formalization efforts have been carried out in game theory since 2006, each focusing on a somewhat different fragment: Vestergaard [34] then Le Roux [30], formalizing Kuhn’s existence of a Nash equilibrium in finite games in extensive form, using Coq; Lescanne et al. [21], studying rationality of infinite games in extensive form, using Coq; Martin-Dorel et al. [23], studying the probability of existence of winning strategies in Boolean finite games, using Coq; Bagnall et al. [4], formalizing well-known results of algorithmic game theory, using Coq; Dittmann [8], proving the positional determinacy of parity games, using Isabelle/HOL; Le Roux et al. [19], proving that a determinacy assumption implies the existence of Nash equilibrium in 2-player games, using Coq and Isabelle/HOL; this result being combined with that of Dittmann, using Isabelle/HOL [20]. Game theory aside, numerous works have been carried out in proof assistants to formalize probability and/or measure theory. Regarding the Coq proof assistant, we can mention the recent works by Affeldt et al. (on information theory based on discrete probability theory [2]; and measure theory based on **MathComp** [1]) and Boldo et al. [5], focusing on Lebesgue’s integral theory.

**Paper outline.** We start by introducing a motivating example, for which the Bayesian

85 approach fails but the Dempster-Shafer approach succeeds. Then, we incrementally present  
 86 the Coq formalization of Bel games, giving successively the main mathematical definitions  
 87 and lemmas and their formal Coq counterpart, throughout the paper: Section 3 presents the  
 88 Dempster-Shafer theory of belief functions; then Section 4 focuses on complete-information  
 89 games (including hypergraphical games) while Section 5 deals with Bel games; then Section 6  
 90 is devoted to our formalization of Howson-Rosenthal’s generalized theorem in the scope of  
 91  $n$ -player Bel games. Finally, Section 7 gives concluding remarks and perspectives.

## 92 **2 Motivating Example: the Murder of Mr. Jones**

93 ► **Example 1** (Inspired by the *Murder of Mr. Jones*, [33]). Two players, named Player 1 and  
 94 Player 2, are independently looking for an association, with either Peter ( $P$ ), Quentin ( $Q$ ),  
 95 or Rose ( $R$ ). The point is that a crime has been committed, for which these three people  
 96 only are suspected. Furthermore, a poor-quality surveillance video allows to estimate that  
 97 there is a 50% chance that the culprit is a man ( $P$  or  $Q$ ), and a 50% chance that it is a  
 98 woman ( $R$ ). As to the interest of the associations, making the deal with an innocent people  
 99 leads to a payoff of \$6k, to be shared between the people making the deal (that is, \$2k or  
 100 \$3k depending on if the players choose the same partner or not); associating with the culprit  
 101 produces no payoff (\$0k). Moreover, Player 1 is investigating about  $P$  and will know whether  
 102 he is guilty before making the decision. Similarly, Player 2 will know whether  $R$  is guilty.

103 The Bayesian approach (claiming any knowledge shall be described by a single subjective  
 104 probability) is not well-suited here. Assume Player 1 learns that  $P$  is innocent. It should  
 105 not impact the evidence of 50% chance per sex, so the probability of guilt should become  
 106  $1/2$  for  $Q$  and  $1/2$  for  $R$ . However, in a purely Bayesian view, a prior probability must be  
 107 made explicit (e.g. by equiprobability assumption:  $1/4$  for  $P$ ,  $1/4$  for  $Q$  and  $1/2$  for  $R$ ), so  
 108 after conditioning the posterior probability would not give 50% chance per sex anymore:  
 109 equiprobability and conditioning “given  $\neg P$ ” yields  $1/3$  for  $Q$  and  $2/3$  for  $R$ : learning that  
 110  $P$  is innocent would increase the odds against  $R$ ! In the sequel, we will reuse this example  
 111 to highlight how the framework of belief functions better captures uncertain knowledge.

## 112 **3 Formalization of Belief Functions for Mono-Agent Decision Making**

113 Modelling mono-agent decision making under uncertainty involves three main tasks. First,  
 114 knowledge has to be expressed in a well-suited representation, encoding what is known  
 115 without making extra assumptions. Then, if the agent may learn or observe some event  
 116 before the decision, one shall identify the relevant conditioning rule. Finally, the agent’s  
 117 preferences must be captured by a compatible decision rule. In this work, we focus on  
 118 real-valued utility-based decision rules, which evaluate uncertain outcomes so that the agent  
 119 prefers outcomes with a bigger score. For example, modelling well-known variable phenomena  
 120 can perfectly be captured in a probabilistic setting: a probability represents the variability;  
 121 conditional probability updates knowledge; and preferences over uncertain outcomes may be  
 122 captured by expected utility. Still, this approach may be unsuccessful to model other kinds  
 123 of uncertainty.

124 In the sequel, we rely on belief function theory, which generalizes probability theory  
 125 and enables capturing both variability and ignorance. In this section, we focus on a single  
 126 decision maker, while the material from Section 4 will deal with multi-agent decision making.

### 127 3.1 Belief functions

128 The theory of belief functions is a powerful toolset from decision theory and statistics. It  
 129 encompasses two distinct approaches for reasoning under uncertainty: the Dempster-Shafer  
 130 theory of evidence (DS) [7, 31] and the upper-lower probability theory (ULP) [7, 35]. Both  
 131 approaches consider a finite set of possible “states of the world”  $\Omega = \{\omega_1, \dots, \omega_n\}$ , one of  
 132 which being the actual state of the world  $\omega^*$ , and three functions  $m$ , Bel and Pl :  $2^\Omega \rightarrow [0, 1]$   
 133 which all map subsets of  $\Omega$  to real numbers. Those functions are deducible one from another.  
 134 In the DS theory, the mass function  $m$  is the basic knowledge about the world ( $m(A)$  is  
 135 the part of belief supporting the evidence  $\omega^* \in A$ : it asserts that  $A$  holds but cannot be  
 136 refined into smaller claims). The non-additive continuous measures Bel and Pl indicate how  
 137 much a proposition is implied by (resp. is compatible with) the knowledge. By contrast, the  
 138 ULP theory suppose that there is an unknown probability  $\Pr^*$  which is bounded by Bel and  
 139 Pl:  $\forall A, \text{Bel}(A) \leq \Pr^*(A) \leq \text{Pl}(A)$ ; then  $m$  is just a concise representation of the family of  
 140 compatible probabilities.

141 Example 1 can be understood in both theories. In the DS theory,  $m(\{P, Q\}) = m(\{R\}) =$   
 142  $1/2$  directly encodes the given evidences. In the ULP theory, one rather considers the family  
 143 of probabilities which satisfy  $\Pr(\{P, Q\}) = \Pr(\{R\}) = 1/2$  (see Table 1).

$A \subseteq \Omega$	$\emptyset$	$\{P\}$	$\{Q\}$	$\{R\}$	$\{P, Q\}$	$\{P, R\}$	$\{Q, R\}$	$\{P, Q, R\}$
$m(A)$	0	0	0	0.5	0.5	0	0	0
$\text{Bel}(A)$	0	0	0	0.5	0.5	0.5	0.5	1
$\text{Pl}(A)$	0	0.5	0.5	0.5	0.5	1	1	1
$\text{Pr}_x(A)$	0	$x$	$0.5 - x$	0.5	0.5	$0.5 + x$	$1 - x$	1

144 **Table 1** Prior knowledge from Example 1 — in the DS theory,  $m$  directly describes the knowledge,  
 145 in the ULP theory,  $m$  describes a family of probability measures  $(\text{Pr}_x)_{x \in [0, 0.5]}$ .

146 In this work, we follow the DS approach and we base our formalization on the mass  
 147 function  $m$ . To do so, we use the Coq proof assistant, relying on the MathComp library [22]  
 148 that provides a comprehensive formalization of finite sets, functions with finite support  
 149 (endowed with decidable equality) as well as big operators.<sup>1</sup>

150 **► Definition 1** (Frame of discernment). *A frame of discernment is a finite set  $\Omega$ , representing*  
 151 *the possible states of the world. One of them is the actual state of the world  $\omega^*$ .*

152 **► Definition 2** (Events). *A set  $A \subseteq \Omega$  is an event which represents the proposition “ $\omega^* \in A$ ”.*  
 153 *All set functions we consider ( $m$ , Bel, Pl) map events to real numbers within  $[0, 1]$ .*

154 The set  $\Omega$  and the carrier of  $m$ , Bel and Pl are formalized in MathComp by asserting:

155 **Variable** ( $\Omega : \text{finType}$ ) ( $\mathbb{R} : \text{realFieldType}$ ).

156 **► Definition 3** (Basic probability assignment). *A basic probability assignment (bpa), a.k.a.*  
 157 *mass function, is a set-function  $m : 2^\Omega \rightarrow [0, 1]$  such that*

$$158 \quad m(\emptyset) = 0 \quad \text{and} \quad \sum_{A \subseteq \Omega} m(A) = 1 \quad \text{and} \quad \forall A \subseteq \Omega, m(A) \geq 0. \quad \text{Formally:} \quad (1)$$

159

<sup>1</sup> MathComp notations:  $\{\text{set } X\}$  denotes finite sets over  $(X : \text{finType})$ ,  $A : \&B = A \cap B$ ,  $A : |B = A \cup B$ ,  
 $\sim : A = A^c$ ,  $\text{set0} = \emptyset$ ;  $\{\text{ffun } X \rightarrow Y\}$  denotes the type of finite support functions from  $(X : \text{finType})$   
 to  $(Y : \text{Type})$ ; and for any  $(T : X \rightarrow \text{Type})$ ,  $\{\text{ffun forall } x : X, T \ x\}$  denotes the type of finite  
 support functions with a dependently-typed codomain, mapping any  $(x : X)$  to an element of  $(T \ x)$ .

```

160
161 Definition bpa_axiom (m : {ffun {set W} → R}) :=
162   [ && m set0 == 0, \sum_A m A == 1 & [∀A, m A >= 0] ].
163
164 Structure bpa := { bpa_val :> {ffun {set W} → R} ; bpa_ax : bpa_axiom bpa_val }.

165 ▶ Definition 4 (Belief function, plausibility measure). Given a bpa m over  $\Omega$ , the associated
166 belief function  $\text{Bel} : 2^\Omega \rightarrow [0, 1]$  and plausibility measure  $\text{Pl} : 2^\Omega \rightarrow [0, 1]$  are defined by:
167   
$$\text{Bel}(A) = \sum_{B \subseteq A} m(B) \quad \text{and} \quad \text{Pl}(A) = \sum_{B \cap A \neq \emptyset} m(B).$$
 Formally:
168
169 Definition Bel (m : bpa) : {set W} → R := fun A => \sum_(B | B \subset A) m B.
170 Definition Pl (m : bpa) : {set W} → R := fun A => \sum_(B | B :&: A != set0) m B.
171

172 ▶ Proposition 1 (Duality). For any  $A \subseteq \Omega$ ,  $\text{Pl}(A) = 1 - \text{Bel}(A^c)$  and  $\text{Bel}(A) = 1 - \text{Pl}(A^c)$ .
173
174 ▶ Proposition 2 (Super- and sub-additivity). Bel is super-additive while Pl is sub-additive:
175 for disjoint sets  $A, B \subseteq \Omega$ ,  $\text{Bel}(A \cup B) \geq \text{Bel}(A) + \text{Bel}(B)$  and  $\text{Pl}(A \cup B) \leq \text{Pl}(A) + \text{Pl}(B)$ .
176
177 ▶ Proposition 3 (Bounds).  $0 = \text{Bel}(\emptyset) = \text{Pl}(\emptyset) \leq \text{Bel}(A) \leq \text{Pl}(A) \leq \text{Bel}(\Omega) = \text{Pl}(\Omega) = 1$ , for
178 any  $A \subseteq \Omega$ . These propositions have been formally proved, with the following statements:
179
180 Lemma BelE (m : bpa) (A : {set W}) : Bel m A = 1 - Pl m (~:A).
181 Lemma Bel_super m A B : [disjoint A & B] → Bel m (A :|: B) >= Bel m A + Bel m B.
182 Lemma Bel0 (m : bpa) : Bel m set0 = 0. (* and likewise for Pl *)

183
184 ▶ Definition 5 (Focal elements, focal set). Given a bpa m over  $\Omega$ , any subset  $A \subseteq \Omega$  with a
185 non-zero mass  $m(A)$  is called focal element, and the set of focal elements of m is called the
186 focal set of m and denoted by  $\mathcal{S}_m$ . Formally:
187
188 Definition focal_element (m : bpa) : pred {set W} := fun A => m A > 0.
189 Definition focalset (m : bpa) : {set {set W}} := [set A | focal_element m A].

190
191 ▶ Proposition 4 (Focal elements, focal set). Given a bpa m over  $\Omega$ , Definition 4 can
192 straightforwardly be rephrased by rewriting the sums over the focal set:
193
194 
$$\text{Bel}(A) = \sum_{\substack{B \in \mathcal{S}_m \\ B \subseteq A}} m(B) \quad \text{and} \quad \text{Pl}(A) = \sum_{\substack{B \in \mathcal{S}_m \\ B \cap A \neq \emptyset}} m(B).$$
 Formally:
195
196 Lemma Bel_focalsetE m A : Bel m A = \sum_(B in focalset m | B \subset A) m B.
197 Lemma Pl_focalsetE m A : Pl m A = \sum_(B in focalset m | B :&: A != set0) m B.

198
199 Next, we recall a standard “complexity definition” about belief functions, that will prove
200 useful to characterize probability measures:
201
202 ▶ Definition 6 ( $k$ -additivity). Given a bpa m over  $\Omega$  and  $k \in \mathbb{N}^*$ , we say that the bpa m is
203  $k$ -additive iff the cardinality of its focal elements is at most  $k$ . Formally:
204
205 Definition k_additivity m := \max_(B in focalset m) #|B|.

206
207 ▶ Proposition 5 (Probability measure). Given a bpa m over  $\Omega$ , if m is 1-additive, i.e. if all
208 focal elements are singletons, then  $\text{Bel} = \text{Pl}$  is a discrete probability measure, associated with
209 the distribution  $x \in \Omega \mapsto m(\{\omega\})$ . Formally:
210
211 Structure proba := { proba_val :> bpa ; proba_ax : k_additivity proba_val == 1 }.
212 Lemma PrE (m : proba) (A : {set W}) : Bel m A = Pl m A.
213 Definition dist (m : proba) := fun w => m [set w]. (*[set w] corresponds to {w} *)
214 Lemma Pr_distE (m : proba) A : Bel m A = \sum_(w in A) dist m w.

```

### 214 3.2 Conditioning in the Belief Function Theory

215 Conditioning is the operation that captures knowledge revision (fact learning) as well as  
 216 focusing (hypothesis) [10, 9, 12]. By turning a prior bpa into a posterior “given an event  
 217  $C$ ”, one updates the knowledge so it now asserts that  $C$  is certain. Several conditioning  
 218 rules for belief functions have been proposed, depending on the DS or ULP interpretation  
 219 (cf. Section 3.1) and on the kind of update it involves. Starting from the same prior bpa,  
 220 they yield distinct posteriors—they indeed capture distinct operations.

221 Before dealing with conditional events  $(\cdot | C)$ —read “given  $C$ ”—a precondition happens  
 222 to be necessary: on the technical side, it avoids division-by-zero, and on the semantics  
 223 side, it means one cannot learn that an impossible event holds. Since the definition of  
 224 this precondition is specific to each conditioning rule, we abstract it away in the form of a  
 225 **revisable** predicate, which indicates whether an event can be assumed.

226 ► **Definition 7** (Conditioning). *Given a bpa  $m$ , a predicate  $\text{revisable}_m : \Omega \rightarrow \{1, 0\}$  and an*  
 227 *event  $C \subseteq \Omega$  such that  $\text{revisable}_m(C)$  holds, a conditioning turns  $m$  into the bpa  $m(\cdot |_{\text{cond}} C)$*   
 228 *such that  $\text{Bel}(C^c |_{\text{cond}} C) = 0$ . Formally:*

```
229 Definition conditioning_axiom (revisable : bpa → pred {set W})
230                               (cond : ∀ m C, revisable m C → bpa) :=
231   ∀ m C (Hrev : revisable m C), Bel (cond m C Hrev) (~:C) = 0.
```

234 In other words, assuming the event  $C$  is revisable implies that if one learns that  $C$  holds,  
 235 then one also learns that no evidence for the complement can hold. Next, we formalize  
 236 a **conditioning** structure that encapsulates the **revisable** predicate, the conditioning  
 237 algorithm itself—which turns a **revisable** prior in its posterior “given  $C$ ”—and a proof of  
 238 the **conditioning\_axiom**:

```
239 Structure conditioning := { revisable : bpa → pred {set W} ;
240                             cond_val :> ∀ m C, revisable m C → bpa ;
241                             cond_ax : conditioning_axiom cond_val }.
```

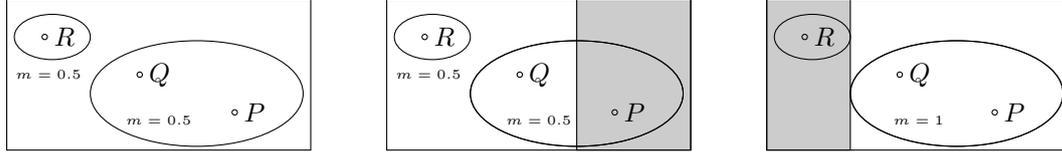
244 The most common conditioning is the so-called Dempster’s conditioning [7], which captures  
 245 knowledge revision (i.e., fact learning). In the DS framework, it is understood as a transfer  
 246 of parts of beliefs: learning that  $C$  holds,  $m(B)$  is transferred to  $B \cap C$  if it is not empty, or  
 247 discarded otherwise (then the posterior has to be renormalized due to Equation (1)). That is,  
 248 the evidence now concerns  $B \cap C$ , the only possible states of the world “given that  $C$  holds”.  
 249 In the ULP framework, it is understood as a max-likelihood conditioning: the posterior  
 250 probability family delimited by  $\text{Bel}(\cdot | C)$  and  $\text{Pl}(\cdot | C)$  is the conditioning of those prior  
 251 probabilities which assign the maximal probability to event  $C$ , that we now know for sure.

252 ► **Definition 8** (Dempster’s conditioning). *For any bpa  $m$  and any event  $C$  such that  $\text{Pl}(C) \neq 0$ ,*  
 253 *Dempster’s conditioning defines the bpa:  $m(A |_D C) = \sum_{B \cap C = A \neq \emptyset} m(B) / \text{Pl}(C)$ .*

```
254 Definition Dempster_revisable m C := Pl m C != 0.
255 Definition Dempster_fun (m : bpa) (C : {set W}) := [ffun A : {set W} =>
256   if A == set0 then 0
257   else \sum_(B : {set W} | (B \in focalset m) && (B :&: C == A)) m B / Pl m C].
258 Program Definition Dempster_cond m C (Hrev : Dempster_revisable m C) : bpa :=
259   { | bpa_val := Dempster_fun m C ; bpa_ax := _ | }.
260 Program Definition Dempster_conditioning : conditioning :=
261   { | cond_val := Dempster_cond ; cond_ax := _ | }.
```

264 ► **Example 2** (Knowledge revision, follow-up of Example 1/Table 1). Dempster’s conditioning is  
 265 the conditioning approach fitting our example (see [9] for details). Suppose e.g. the murderer

266 is  $Q$ ; Player 1 learns  $\omega^* \notin \{P\}$ , i.e.,  $\omega^* \in \{Q, R\}$ . From this viewpoint, the evidence  
 267 concerning men now only concerns  $Q$ : the knowledge becomes  $m(\{Q\}) = m(\{R\}) = 0.5$   
 268 (Fig. 1, center). Player 2 learns  $\omega^* \notin \{R\}$ , i.e.,  $\omega^* \in \{P, Q\}$ . From this viewpoint, the  
 evidence about women is discarded: the knowledge becomes  $m(\{P, Q\}) = 1$  (Fig. 1, right).



269 **Figure 1** Prior (left) and posteriors given  $\{Q, R\}$  (center) and given  $\{P, Q\}$  (right). White and  
 gray areas denote possible and impossible events – circles denote focal elements.

270 **Proposition 6** (Dempster's conditioning, Pl). *For any bpa  $m$  and any event  $C$  such that*  
 271  $\text{Pl}(C) \neq 0$ , *we have  $\text{Pl}(A \mid_D C) = \text{Pl}(A \cap C) / \text{Pl}(C)$ . Formally:*

272 **Lemma** Dempster\_condE m C (Hrev : revisable m C) :  
 273  $\forall A, \text{Pl} (\text{Dempster\_conditioning } m \ C \ \text{Hrev}) \ A = \text{Pl } m \ (A \ \&: \ C) / \text{Pl } m \ C.$   
 274  
 275

276 Two other rules have been proposed and called strong (resp. weak) conditioning [28]; the  
 277 former, also known as geometrical conditioning [32], is another rule capturing knowledge  
 278 revision; the latter is seldom used since it yields non-intuitive results (e.g., it may happen  
 279 that  $\text{Bel}(C \mid C) < 1$ ). We also formalize these two rules below.

280 **Definition 9** (Strong conditioning). *For any bpa  $m$  and any event  $C$  such that  $\text{Bel}(C) \neq 0$ ,*  
 281 *the weak conditioning is defined by the bpa  $m(A \mid_S C) = m(A) / \text{Bel}(C)$  if  $A \subseteq C$ , 0 otherwise.*

282 **Definition** Strong\_revisable m C := Bel m C != 0.  
 283 **Definition** Strong\_fun (m : bpa) (C : {set W}) := [ffun A : {set W} =>  
 284 if (A != set0) && (A \subset C) then m A / Bel m C else 0].  
 285 **Program Definition** Strong\_cond m C (Hrev : Strong\_revisable m C) : bpa :=  
 286 { | bpa\_val := Strong\_fun m C ; bpa\_ax := \_ | }.  
 287 **Program Definition** Strong\_conditioning : conditioning :=  
 288 { | cond\_val := Strong\_cond ; cond\_ax := \_ | }.  
 289  
 290

291 **Proposition 7** (Strong conditioning, Bel). *For any bpa  $m$  and any event  $C$  such that*  
 292  $\text{Bel}(C) \neq 0$ , *we have  $\text{Bel}(A \mid_S C) = \text{Bel}(A \cap C) / \text{Bel}(C)$ . Formally:*

293 **Lemma** Strong\_condE m C (Hrev : revisable m C) :  
 294  $\forall A, \text{Bel} (\text{Strong\_conditioning } m \ C \ \text{Hrev}) \ A = \text{Bel } m \ (A \ \&: \ C) / \text{Bel } m \ C.$   
 295  
 296

297 **Definition 10** (Weak conditioning). *For any bpa  $m$  and any event  $C$  such that  $\text{Pl}(C) \neq 0$ ,*  
 298 *the weak conditioning is defined by the bpa:  $m(A \mid_W C) = m(A) / \text{Pl}(C)$  if  $A \cap C \neq \emptyset$ ,*  
 299 *0 otherwise. Formally:*

300 **Definition** Weak\_revisable m C := Pl m C != 0.  
 301 **Definition** Weak\_fun (m : bpa) (C : {set W}) := [ffun A : {set W} =>  
 302 if A \&: C != set0 then m A / Pl m C else 0].  
 303 **Program Definition** Weak\_cond m C (Hrev : Weak\_revisable m C) : bpa :=  
 304 { | bpa\_val := Weak\_fun m C ; bpa\_ax := \_ | }.  
 305 **Program Definition** Weak\_conditioning : conditioning :=  
 306 { | cond\_val := Weak\_cond ; cond\_ax := \_ | }.  
 307  
 308

309 **Proposition 8** (Weak conditioning, Bel). *For any bpa  $m$  and any event  $C$  such that*  
 310  $\text{Pl}(C) \neq 0$ , *we have  $\text{Bel}(A \mid_W C) = (\text{Bel}(A) - \text{Bel}(A \setminus C)) / \text{Pl}(C)$ . Formally:*

311 **Lemma** Weak\_condE m C (Hrev : revisable m C) :  
 312  $\forall A, \text{Bel} (\text{Weak\_conditioning } m \ C \ \text{Hrev}) \ A = (\text{Bel } m \ A - \text{Bel } m \ (A \ \setminus: \ C)) / \text{Pl } m \ C.$   
 313

### 3.3 Decision Making with Belief Functions

Consider a single agent decision involving several actions; let  $A$  denote the set of all these actions. Also, assume that the outcome of choosing any  $a \in A$  is not certain: it may lead to several outcomes depending on the actual state of the world  $\omega^*$ . The agent's preferences on outcomes (which are left implicit here) are expressed by a real-valued utility function  $u : A \times \Omega \rightarrow \mathbb{R}$ :  $u(a, \omega) > u(a', \omega')$  would mean the agent prefers the outcome of  $a$  when  $\omega^* = \omega$  to the outcome of  $a'$  when  $\omega^* = \omega'$ . For any action  $a$ , let  $u_a : \Omega \rightarrow \mathbb{R}$  denote the partial application of  $u$ :  $u_a$  provides the utility of  $a$  depending on the state of the world  $\omega$ .

Preferences under uncertainty are then defined on  $u_a$ 's: a relation  $u_a \succ u_{a'}$  would encode the fact the agent prefers  $a$  to  $a'$ . In a probabilistic setting, it is meaningful to consider  $u_a$ 's expectation w.r.t. the probability (hence the name *expected utility*). Using bpa's, several approaches were defined, each modelling various preferences when facing ignorance. In [29], we analyzed three standard functions that generalize expected utility. They provide real values, and thus lead to completely ordered preferences over actions (since every two actions are directly comparable from their score). We denoted them CEU, JEU, and TBEU, respectively, for Choquet-, Jaffray-, and Transferable Belief-Expected Utility. We have shown they are all expressible as the integration of a particular  $\varphi_{u_a}^{\text{XEU}}$  function (resp.  $\varphi_{u_a}^{\text{CEU}}$ ,  $\varphi_{u_a}^{\text{JEU}}$ , and  $\varphi_{u_a}^{\text{TBEU}}$ ) over the powerset  $2^\Omega$ . Those  $\varphi_{u_a}^{\text{XEU}}$  functions are themselves parametrized by  $u_a = \omega \mapsto u(a, \omega)$ , that is, by the utility function when  $a$  is chosen. As a result, these three scoring functions can be captured by instances of a single higher-order function, which we named XEU.

► **Definition 11** (Generalized expected utility). *For any bpa  $m$ , any utility function  $u : A \times \Omega \rightarrow \mathbb{R}$ , and any  $a \in A$ , let us pose  $u_a = \omega \mapsto u(a, \omega)$ . Let  $\varphi : (\Omega \rightarrow \mathbb{R}) \rightarrow (2^\Omega \rightarrow \mathbb{R})$  be a parameter function. We then consider the following generalized expected utility of  $a$ :*

$$\text{XEU}(m)(\varphi(u_a)) = \sum_{B \in \mathcal{S}_m} m(B) \times \varphi(u_a)(B). \quad \text{Formally:}$$

```

341 Definition XEU (m : bpa) (phi_u_a : {ffun {set W} -> R}) : R :=
342   \sum (B in focalset m) m B * phi_u_a B.
343

```

Let us review these  $\varphi^{\text{XEU}}$  functions, their underlying intuition and formal definition in Coq.

A very common scoring function for belief functions is the Choquet discrete integral (CEU). It models a somehow pessimistic agent. In the ULP interpretation, Bel and Pl delimit a family of probabilities; the CEU computes the minimal expected utility that the family allows. In the DS interpretation, each mass is an evidence supporting an event, for which the CEU only consider its worst-case utility if the considered choice is made.

► **Definition 12** (Choquet expected utility). *For any bpa  $m$ , any utility function  $u : A \times \Omega \rightarrow \mathbb{R}$  and any action  $a \in A$ , the Choquet expected utility of  $u_a : \Omega \rightarrow \mathbb{R}$  is:*

$$\text{CEU}(m)(u_a) = \sum_{B \in \mathcal{S}_m} m(B) \times \min_{\omega \in B} u_a(\omega) = \text{XEU}(m)(\varphi^{\text{CEU}}(u_a)),$$

with  $\varphi^{\text{CEU}}(u_a)(B) = \min_{\omega \in B} u_a(\omega)$ .

This expression is a weighted sum indexed by the set of focal elements, which is nonempty: using the min operator is legit. Formally, the functions  $\varphi^{\text{CEU}}$  and CEU are defined as follows:

```

357 Definition fCEU (u_a : W -> R) : {set W} -> R :=
358   fun B => match minS u_a B with Some r => r | None => 0 end.
359 Definition CEU (m : bpa) (u_a : W -> R) := XEU m (fCEU u_a).
360

```

Another rule, axiomatized by Jaffray [16, 17], is a kind of Hurwicz criterion (i.e., a linear combination over the min. and max. utility reached for each focal element). The parameter coefficients make it possible to locally modulate the pessimism of the modelled agent.

365 ▶ **Definition 13** (Jaffray expected utility). *For any bpa  $m$ , any utility function  $u : A \times \Omega \rightarrow \mathbb{R}$*   
 366 *and any action  $a \in A$ , the Jaffray expected utility of  $u_a : \Omega \rightarrow \mathbb{R}$  is parameterized by a family*  
 367 *of coefficients  $\alpha_{(x_*, x^*)} \in [0, 1]$  for each possible utility values  $x_* \leq x^*$ . For any  $B \neq \emptyset$ , let us*  
 368 *pose  $B_* = \min_{\omega \in B} u_a(\omega)$  and  $B^* = \max_{\omega \in B} u_a(\omega)$ . The Jaffray expected utility of  $u_a$  is:*

$$369 \quad \text{JEU}^\alpha(m)(u_a) = \sum_{B \in \mathcal{S}_m} m(B) \times (\alpha_{(B_*, B^*)} \times B_* + (1 - \alpha_{(B_*, B^*)}) \times B^*) = \text{XEU}(m)(\varphi^{\text{JEU}^\alpha}(u_a)),$$

370 *with  $\varphi^{\text{JEU}^\alpha}(u_a)(B) = \alpha_{(B_*, B^*)} \times B_* + (1 - \alpha_{(B_*, B^*)}) \times B^*$ . Formally:*

```
371 Definition fJEU (alpha : R -> R -> R) (u_a : W -> R) : {set W} -> R :=
372   fun B => match minS u_a B, maxS u_a B with
373     | Some rmin, Some rmax => let alp := alpha rmin rmax in alp * rmin + (1-alp) * rmax
374     | _, _ => 0   end.
375 Definition JEU alpha (m : bpa) (u_a : W -> R) := XEU m (fJEU alpha u_a).
```

378 Last, in the Transferable Belief Model [33], the decision rule is made by recovering a  
 379 “pignistic” probability distribution<sup>2</sup> BetP (namely, a probability that serves only for the  
 380 choice) at the very moment where the decision is made (so, the equiprobability assumption is  
 381 made, but after conditionings, if any). The score of an action is then the expected utility w.r.t.  
 382 BetP :  $\omega \mapsto \sum_{\substack{B \in \mathcal{S}_m \\ \omega \in B}} m(B)/|B|$ , that we show to be equivalent to the following definition.

383 ▶ **Definition 14** (Transferable Belief Model expected utility). *For any bpa  $m$ , any utility*  
 384 *function  $u : A \times \Omega \rightarrow \mathbb{R}$  and any action  $a \in A$ , the TBEU of  $u_a : \Omega \rightarrow \mathbb{R}$  is defined by:*

$$385 \quad \text{TBEU}(m)(u_a) = \sum_{B \in \mathcal{S}_m} m(B) \times \sum_{\omega \in B} u_a(\omega) / |B| = \text{XEU}(m)(\varphi^{\text{TBEU}}(u_a))$$

386 *with  $\varphi^{\text{TBEU}}(u_a)(B) = \sum_{\omega \in B} u_a(\omega) / |B|$ . Formally:*

```
387 Definition fTBEU (u_a : W -> R) := fun B => \sum_w (w in B) u_a w / #|B|:R.
388 Definition TBEU (m : bpa) (u_a : W -> R) := XEU m (fTBEU u_a).
```

391 ▶ **Proposition 9.** *CEU, JEU, and TBEU all generalize the expected utility criterion: if the bpa*  
 392  *$m$  is 1-additive (i.e., it is a discrete probability measure), all these instances of XEU yield*  
 393 *the same value as that of the expected utility:*

```
394 Lemma CEU_EU (p : proba) u_a : CEU p u_a = \sum_w dist p w * u w.
395 Lemma JEU_EU alpha (p : proba) u_a : JEU alpha p u_a = \sum_w dist p w * u w.
396 Lemma TBEU_EU (p : proba) u_a : TBEU p u_a = \sum_w dist p w * u w.
```

399 In these formal proofs, the key ingredient is the fact that the criteria satisfy the natural  
 400 property that  $\forall u, \forall \omega \in \Omega, \varphi(u)(\{\omega\}) = u(\omega)$ .

## 401 4 Formalization of Several Classes of Games of Complete Information

402 Game theory is a subdomain of multi-agent decision making [24, 25]. In this paper, we focus  
 403 on simultaneous games, in which *players* make their choice (called *action* or *pure strategy*)  
 404 without knowing others’ choices in advance; the outcome of an action depends on the choices  
 405 of other agents. A typical problem amounts to identifying which actions are relevant from  
 406 the viewpoint of a player, assuming others don’t cooperate but strive to increase their own  
 407 utility. In this section, we consider situations where there is no uncertainty.

<sup>2</sup> The names “pignistic” and BetP are references to classical Bayesian justification in decision theory, where both utilities and beliefs are elicited by considering limits of agent’s agreement to a panel of bets.

## 4.1 Games of Complete Information

► **Definition 15** (Game of complete information). A *CGame* is a tuple  $G = (I, (A_i, u_i)_{i \in I})$  where  $I$  is a finite set of players; for each Player  $i$ ,  $A_i$  is the set of their actions;  $u_i : A \rightarrow \mathbb{R}$  is an utility function, assigning an utility value to each “action profile”, i.e., a vector of actions, also called “pure strategy profile”  $a = (a_1, \dots, a_n) \in A = A_1 \times \dots \times A_n$ . Player  $i$  prefers the outcome of profile  $a$  to that of  $a'$  iff  $u_i(a) > u_i(a')$ .

We formalize such “profiles-for-CGames” ( $a \in \prod_{i \in I} A_i$ ) using MathComp’s dependently-typed finite support functions, hence:

**Definition** `cprofile` ( $I : \text{finType}$ ) ( $A : I \rightarrow \text{eqType}$ ) := `{ffun  $\forall i : I, A i$ }`.

**Definition** `cgame` ( $I : \text{finType}$ ) ( $A : I \rightarrow \text{eqType}$ ) := `cprofile A  $\rightarrow I \rightarrow \mathbb{R}$` .

One of the most prominent solution concept in game theory is that of Nash equilibrium [26]:

► **Definition 16** (Nash equilibrium). A pure Nash equilibrium is a profile such that no player has any incentive to “deviate”. For any pure strategy profile  $a$  and any Player  $i$ , let  $a_{-i}$  be the restriction of  $a$  to the actions of Players  $j \neq i$ ,  $a'_i$  an action of Player  $i$ , then  $a'_i.a_{-i}$  denotes the profile  $a$  where the strategy of Player  $i$  has been switched to  $a'_i$  (called `change_strategy a a'_i` in Coq). A profile  $a$  is a pure Nash equilibrium iff  $\forall i, \forall a'_i, u_i(a) \not< u_i(a_i.a_{-i})$ :

**Definition** `change_strategy` ( $p : \text{cprofile } A$ ) ( $i : I$ ) ( $a'_i : A i$ ) : `cprofile A`

**Definition** `Nash_equilibrium` ( $G : \text{cgame}$ ) ( $a : \text{cprofile } A$ ) : `bool` :=

`[ $\forall i : I, [\forall a'_i : A i, \sim (G a i < G (\text{change\_strategy } a a'_i) i)]]$ ].`

► **Example 3.** Consider Example 1 anew; suppose one knows  $P$  is the murderer. The situation is captured by the CGame  $G = (I, (A_i, u_i)_{i \in I})$  where  $I = \{1, 2\}$  is the set of players,  $A_i = \{P_i, Q_i, R_i\}$  the set of actions of Player  $i$  (choosing  $P$ ,  $Q$  or  $R$ ) and the  $u_i$ ’s of Table 2.

	$P_2$	$Q_2$	$R_2$
$P_1$	(0, 0)	(0, 3)	(0, 3)
$Q_1$	(3, 0)	(2, 2)	(3, 3)
$R_1$	(3, 0)	(3, 3)	(2, 2)

■ **Table 2** Utility functions of Example 3 (it is known that  $P$  is the murderer). The pair  $(u_1(a_1.a_2), u_2(a_1.a_2))$  is read at the intersection of line  $a_1$  and column  $a_2$ .

Here, both  $(Q_1, R_2)$  and  $(R_1, Q_2)$  are Nash equilibria.

When there is some variability regarding action choices (e.g., for repeated games), it is meaningful to look for mixed strategies. A mixed strategy  $\rho_i$  of Player  $i$  is a probability over  $A_i$ , and a mixed strategy profile  $\rho = (\rho_1, \dots, \rho_n)$  is a vector of mixed strategies:

**Definition** `mixed_cprofile` := `cprofile (fun i  $\Rightarrow$  [eqType of proba R (A i)])`.

A mixed strategy profile  $\rho$  defines a probability over the set of pure strategy profiles, namely  $p_\rho(a) = \prod_{i \in I} \rho_i(a_i)$ . We package this data in a `proba` structure:

**Definition** `mk_prod_proba` ( $p : \forall i : X, \text{proba R (A i)}$ ) : `{ffun cprofile A  $\rightarrow$  R}` :=

`[ffun a : cprofile A  $\Rightarrow$  \prod_i dist (p i) (a i)]`.

**Definition** `prod_proba` ( $p : \forall i : I, \text{proba R (A i)}$ ) ( $i0 : I$ ) : `proba R (cprofile A)`.

Last, the utility of a mixed strategy profile is the expected utility w.r.t. the probability over pure strategy profiles, and the notion of Nash equilibrium extends straightforwardly:

**Definition** `ms_util` ( $G : \text{cgame R A}$ ) ( $mp : \text{mixed\_cprofile}$ ) ( $i : I$ ) : `R` :=

`\sum (p : cprofile A) (dist (prod_proba mp witnessI mp) p) * (G p i)`.

**Definition** `ms_Nash_equilibrium` ( $G : \text{cgame R A}$ ) ( $mp : \text{mixed\_cprofile}$ ) : `Prop` :=

`$\forall i$  (si : proba R (A i)),  $\sim$  ms_util G mp i < ms_util G (change_strategy mp si) i`.

456 A standard reduction [18, Def. 4.6.1] amounts to viewing a mixed equilibrium of a game  
 457  $(N, (A_i, u_i)_{i \in N})$  as a pure equilibrium in the mixed extension  $(N, (\mathcal{A}_i, u_i)_{i \in N})$ , where  $\mathcal{A}_i$  is  
 458 the set of mixed strategies over  $A_i$ . Formally:

```
459 Definition mixed_cgame (G:cgame R A) : cgame R (fun i => [eqType of proba R (A i)])
460 := fun mp i => ms_util G mp i.
461
462 Lemma mixed_cgameE G mp i : ms_utility G mp i = (mixed_cgame G) mp i.
463 Lemma ms_NashE (G : cgame R A) (mp : mixed_cprofile) :
464 ms_Nash_equilibrium G mp <=> Nash_equilibrium (mixed_cgame G) mp.
465
```

## 466 4.2 Hypergraphical Games

467 Some games of complete information can be expressed succinctly as hypergraphical games  
 468 [27, 36], where the utility is not defined globally but locally (namely, split in several “local  
 469 games”). This yields a hypergraph, where vertices denotes players and hyperedges denote  
 470 local games. Formally, a hypergraphical game is a tuple  $G = (I, E, (A_i)_{i \in I}, (u_i^e)_{e \in E, i \in e})$ ,  
 471 where  $I$  is the set of players,  $E \subseteq 2^I$  is the set of local games (in any local game  $e =$   
 472  $\{a, b, c, \dots\}$ , Players  $a, b, c, \dots$  are playing),  $A_i$  is the set of actions of Player  $i$  and  $u_i^e : A_e \rightarrow \mathbb{R}$   
 473 is the utility function of Player  $i$  in the local game  $e$  ( $A_e = \prod_{i \in e} A_i$  is the set of local profiles  
 474 related to  $e$ 's players). A hypergraphical game with 2-player local games is called a polymatrix.

475 In our formalization, local games are indexed by the finite type (`localgame : finType`);  
 476 players playing a local game (`lg : localgame`) are those who verify the Boolean predicate  
 477 (`plays_in lg`); `plays_in` thus formalizes  $E$  as a family of sets of players:

```
478 Variables (localgame : finType) (plays_in : localgame -> pred I).
479
480
```

481 For any local game `lg`, local profiles are profiles that involve only players which `plays_in lg`:

```
482 Definition localprof (lg : localgame) :=
483 {ffun vs : {i : I | plays_in lg i}, A (val s)}.
484
```

486 In hypergraphical games, every player chooses one action, and plays it in every local game  
 487 they are involved in. The global utility of a player is the sum of the locally obtained  
 488 utilities:  $u_i(a) = \sum_{\substack{e \in E \\ i \in e}} u_i^e(a_e)$ , where  $a_e \in A_e$  is the restriction of  $a$  to indices of  $e$ . Thus, an  
 489 hypergraphical game is a CGame that is specified by its local utility functions:

```
490 Definition hg_game (u : ∀ lg, localprof lg -> {i : I & plays_in lg i} -> R) : cgame
491 := fun a i => \sum_(s : {lg : localgame | plays_in lg i})
492 u (tag s) [ffun i => a (val i)] (exist _ i (tagged s)).
493
494
```

## 495 5 Bel Games

496 Harsanyi has proposed [13] a model for decision-making situations where players may have  
 497 some uncertainty about other players, their actions, their utility functions, or more generally  
 498 about any parameter of the game. To model such situations, the partially known parameters  
 499 are expressed by so-called types:<sup>3</sup> each Player  $i$  has a set of possible types  $\Theta_i$ . Each type  
 500  $\theta_i \in \Theta_i$  represents a possible parameter describing Player  $i$ 's characteristics and knowledge.  
 501 Every Player  $i$  knows (or learns) their own type  $\theta_i \in \Theta_i$  at the time of choosing an action. It  
 502 may or may not be correlated with other players' types, so it is possible to model players  
 503 that are not aware of other players' type as well as players with some knowledge about them.

<sup>3</sup> Thus, *type* can refer to a type-theory concept or a game-theory one. Context will allow to disambiguate.

504 Harsanyi defined the model of games of incomplete information where players' knowledge  
 505 is given by a subjective probability and preferences agree with the expected utility: the  
 506 so-called class of Bayesian games. In this setting, a probability measure expresses the  
 507 knowledge on type configurations (the frame of discernment being the cartesian product of  
 508 all players' types, that is,  $\Omega = \prod_{i \in N} \Theta_i = \Theta$ ). Games of incomplete information already  
 509 were extended to a possibilistic setting by Ben Amor et al. [3], and we extend it further to  
 510 Belief functions (so generalizing both Bayesian games and possibilistic games) [11, 29].

511 **► Definition 10** (Bel game). *A Bel game [29] is defined by a tuple  $G = (I, (A_i, \Theta_i, u_i)_{i \in I}, m)$ :*  
 512 **■**  *$I$  is the finite set of players;*  
 513 **■**  *$A_i$  is the set of actions of Player  $i$ ;  $\Theta_i$  is the finite set of types of Player  $i$ ;*  
 514 **■**  *$u_i : A \rightarrow \mathbb{R}$  is the utility function of Player  $i$ ; it depends on the joint action*  
 515  *$(a_1, \dots, a_n) \in A := \prod_{i \in I} A_i$  and on the type configuration  $(\theta_1, \dots, \theta_n) \in \Theta := \prod_{i \in I} \Theta_i$ ;*  
 516 **■**  *$m : 2^\Theta \rightarrow [0, 1]$  is a bpa which describes the prior knowledge.*

517 Formally speaking, a Bel game is fully defined by two elements: the bpa (prior knowledge)  
 518 and the utility functions (the players' preferences). This pair is parameterized by three types  
 519  $I$ , the players;  $A$ , the family of actions  $(A_i)_i$ ; and  $T$ , the family of types  $(\Theta_i)_i$ :

```
520 Definition belgame (I : finType) (A : I → eqType) (T : I → finType) :=
521 (bpa R (cprofile T) * (cprofile A → cprofile T → player → R)).
```

524 **► Example 4** (Bel game). We now are able to endow Example 1 with a Bel game  $G =$   
 525  $(I, (A_i, \Theta_i, u_i)_{i \in I}, m)$ . The set of players is  $\{1, 2\}$ , their action sets are  $A_i = \{P_i, Q_i, R_i\}$ .  
 526 Player 1 will learn either that  $P$  is the murderer ( $\omega^* \in \{P\}$ ) or that he is not ( $\omega^* \in \{Q, R\}$ ):  
 527 Player 1's type set is  $\Theta_1 = \{P, \bar{P}\}$ . Similarly, Player 2 will learn either that  $R$  is the murderer  
 528 ( $\omega^* \in \{R\}$ ) or that she is not ( $\omega^* \in \{P, Q\}$ ), so  $\Theta_2 = \{R, \bar{R}\}$ . The knowledge is expressed over  
 529  $\Theta = \Theta_1 \times \Theta_2$ : since  $(P, \bar{R}) \equiv P$ ,  $(\bar{P}, \bar{R}) \equiv Q$ ,  $(\bar{P}, R) \equiv R$  and  $(P, R)$  is impossible, the knowledge  
 is  $m(\{(P, \bar{R}), (\bar{P}, \bar{R})\}) = m(\{(\bar{P}, R)\}) = 0.5$ . Finally, utility functions are given in Table 3.

	$P_2$	$Q_2$	$R_2$		$P_2$	$Q_2$	$R_2$		$P_2$	$Q_2$	$R_2$
$P_1$	(0, 0)	(0, 3)	(0, 3)	$P_1$	(2, 2)	(3, 0)	(3, 3)	$P_1$	(2, 2)	(3, 3)	(3, 0)
$Q_1$	(3, 0)	(2, 2)	(3, 3)	$Q_1$	(0, 3)	(0, 0)	(2, 2)	$Q_1$	(3, 3)	(2, 2)	(3, 0)
$R_1$	(3, 0)	(3, 3)	(2, 2)	$R_1$	(3, 3)	(3, 0)	(2, 2)	$R_1$	(0, 3)	(0, 3)	(0, 0)

530 **■ Table 3** Utility functions of Example 4 for  $\theta = (P, \bar{R})$  (left,  $P$  is the murderer),  $\theta = (\bar{P}, \bar{R})$  (center,  
 531  $Q$  is the murderer) and  $\theta = (\bar{P}, R)$  (right,  $R$  is the murderer). Configuration  $\theta = (P, R)$  can't occur.

531 Since players know their own type before choosing their action, a pure strategy of Player  $i$   
 532 becomes a function  $\sigma_i : \Theta_i \rightarrow A_i$ : having the type  $\theta_i$ , Player  $i$  will play  $\sigma_i(\theta_i) \in A_i$ . Next, a  
 533 strategy profile  $\sigma = (\sigma_1, \dots, \sigma_n)$  is a vector of such functions:

```
534 Definition iprofile I A T := cprofile (fun i => [eqType of {ffun T i → A i}]).
```

537 If the actual type configuration is  $\theta = (\theta_1, \dots, \theta_n)$ , then for any strategy profile  $\sigma$  we denote  
 538 by  $\sigma^\theta = (\sigma_1(\theta_1), \dots, \sigma_n(\theta_n)) \in A$  the action profile that will actually be played:

```
539 Definition proj_iprofile I A T (p : iprofile A T) : cprofile A :=
540 fun theta => [ffun i => p i (theta i)].
```

541 In the following, we denote by  $u_{i,\sigma} : \Theta \rightarrow \mathbb{R}$  the function mapping states of the world  $\omega$  to  
 542 the corresponding utility of  $\sigma$  for Player  $i$ . It is defined by  $u_{i,\sigma}(\theta) = u_i(\sigma^\theta, \theta)$ .

543 In *Bayesian games*, the global utility of a strategy profile  $\sigma$  for Player  $i$  with type  $\theta_i$  is the  
 544 expected utility w.r.t. the conditioned probability distribution “given  $\theta_i$ ”. In *Bel games*, both  
 545 expectation and conditioning have to be made explicit, to properly model agents' preferences  
 546 and knowledge updates. For example, studying a Bel game with Dempster's conditioning and

CEU expectation implies that the utility of a given strategy profile  $\sigma$  for Agent  $i$  with type  $\theta_i$  is  $\sum_{B \subseteq \Omega} m(B \mid_D \theta_i) \times \min_{\theta' \in B} u_i(\sigma^{\theta'}, \theta')$ . Doing so, we need to ensure that conditioning is meaningful and technically possible, that is, the bpa is **revisable** given any type of any player. For the sake of readability, we now introduce two shorthands: **Tn**, representing the set  $\Theta$  gathering all type configurations; and **event\_ti** :=  $\theta_i \mapsto \{\theta' \in \Theta \mid \theta'_i = \theta_i\}$ :

```

549
550
551
552
553
554
555 Notation Tn := [finType of {dffun  $\forall i : I, T i$  }].
556 Definition event_ti i (ti : T i) := [set t : Tn | t i == ti].
557

```

A *proper Bel games*, in which conditioning is safe, shall satisfy the predicate:

```

558
559
560 Definition proper_belgame A T (G : belgame A T) (cond : conditioning R Tn) : bool
561 := [ $\forall i : \text{player}, [\forall ti : T i, \text{revisable cond G.1 (event\_ti ti)}]]$ .
562

```

► **Definition 11** (Utility in a Bel game). *For any Bel game  $G = (I, (A_i, \Theta_i, u_i)_{i \in I}, m)$ , any conditioning **cond** for which  $G$  is proper and any XEU parameter  $\varphi^{\text{XEU}} : (\Theta \rightarrow \mathbb{R}) \rightarrow 2^\Theta \rightarrow \mathbb{R}$ , the utility of the pure strategy profile  $\sigma$  for Player  $i$  having type  $\theta_i \in \Theta_i$ , is the integration of  $u_{i,\sigma} = \theta \mapsto u_i(\sigma^\theta, \theta)$ , i.e.,  $\text{XEU}(m(\cdot \mid_{\text{cond}} \theta_i))(\varphi^{\text{XEU}}(u_{i,\sigma}))$ .*

```

563
564
565
566
567
568 Definition belgame_utility A T (G : belgame A T) (cond : conditioning R Tn)
569 fXEU (HG : proper_belgame G cond) (p : iprofile A T) (i : player) (ti : T i) : R
570 := let kn := cond G.1 (event_ti ti) (is_revisable HG ti) in
571 XEU kn (fXEU (fun t => G.2 (proj_iprofile p t) t i)).
572

```

Also, for Bel games, the definition of Nash equilibrium applies: an **iprofile** is a Nash equilibrium iff no player, whatever is this player's type, has any incentive to deviate:

```

573
574
575
576 Definition BelG_Nash_equilibrium A T (G : belgame A T) (cond : conditioning R Tn)
577 fXEU (H : proper_belgame G cond) (p : iprofile A T) :=
578  $\forall i : I, \forall ti : T i, \forall ai : A i,$ 
579  $\sim (\text{belgame\_utility } u \text{ H } p \text{ ti} < \text{belgame\_utility } u \text{ H } (\text{change\_istrategy } p \text{ ti } ai) \text{ ti})$ .
580

```

► **Example 5** (Utility of a strategy). Let  $\sigma = (\sigma_1, \sigma_2)$  be defined by  $\sigma_1(P) = Q_1, \sigma_1(\bar{P}) = P_1, \sigma_2(R) = Q_2, \sigma_2(\bar{R}) = R_2$ .  $\sigma$  is a pure strategy asserting that Player 1 will choose  $Q$  when learning that  $P$  is the murderer, and choose  $P$  otherwise, and that Player 2 will choose  $Q$  when learning that  $R$  is the murderer, and choose  $R$  otherwise.

Considering Dempster's conditioning, the Choquet expected utility of  $\sigma$  for Player 1 with type  $\bar{P}$  is the integration of  $\varphi^{\text{CEU}}(u_{i,\sigma})$  w.r.t. the posterior bpa  $m(\cdot \mid \bar{P})$ . Recall Example 2, the posterior bpa "given  $\bar{P}$ " has two focal elements:  $\{Q\}$  and  $\{R\}$ , both with mass 1/2. Considering type configurations, those focal elements are  $\{(\bar{P}, \bar{R})\}$  and  $\{(\bar{P}, R)\}$ .

$$\begin{aligned}
 \text{XEU}(m(\cdot \mid_D \bar{P}))(\varphi^{\text{CEU}}(u_{1,\sigma})) &= \sum_{B \in \mathcal{S}_{m(\cdot \mid_D \bar{P})}} m(B \mid_D \bar{P}) \times \min_{\theta \in B} u_1(\sigma^\theta, \theta) \\
 &= 0.5 \times u_1((P_1, Q_2), (\bar{P}, R)) + 0.5 \times u_1((P_1, R_2), (\bar{P}, \bar{R})) = 3.
 \end{aligned}$$

One may check that for every player and type,  $\sigma$ 's CEU equals 3, the best possible score. Since no player, whatever is their type, has incentive to deviate,  $\sigma$  is a Nash equilibrium.

## 6 Howson-Rosenthal-like transforms

Howson–Rosenthal's theorem asserts the correctness of a transform, which casts a 2-player Bayesian game into an equivalent polymatrix game (of complete information) [15]. Bayesian games thus benefit from both theoretical and algorithmic results of classical game theory.

In the following, we formally define and prove correct three Howson–Rosenthal-like transforms that we have devised in previous work [11, 29]. All these transforms cast  $n$ -player Bel games into hypergraphical games; the games so obtained all have the same utility values,

though different hypergraphs. These transforms can be applied safely, depending on the conditioning and on the decision rule (cf. Table 4): Dempster’s conditioning is hard-coded into the Direct transform while the TBM transform’s low complexity comes from properties of the distribution BetP considered by the TBEU.

Transform	Conditioning	XEU	Space	Time
Direct transform	Dempster’s c.	any	$O(k \times \text{Size}(G)^k)$	$O(k \times \text{Size}(G)^k)$
Conditioned transform	any	any	$O(k \times \text{Size}(G)^k)$	$O(k \times \text{Size}(G)^k)$
TBM transform	any	TBEU	$O(k \times \text{Size}(G))$	$O(\text{Size}(G))$

**Table 4** Transforms, conditioning and XEU they are suited for, and their worst-case complexity w.r.t. the  $k$ -additivity of the bpa and the size of the input Bel game (taken from [29]).

The three transforms all follow the same approach: starting from a Bel game  $G$ , they build the equivalent hypergraphical game  $\tilde{G}$ , with pairs  $(i, \theta_i)$  as “abstract” players (i.e.,  $\tilde{G}$ ’s vertices), denoting every type of every player of  $G$ . The local games correspond to focal elements, so Player  $(i, \theta_i)$  plays in a local game  $\text{lg}$  iff the type  $\theta_i$  is possible in the corresponding focal element. Doing so, we benefit from the hypergraphical game structure to compute an XEU (recall that global utility is the sum of local utilities and that the XEU value is the weighted sum of utilities w.r.t. focal elements). For all those transforms, let  $(G : \text{belgame } A \ T)$  be the input Bel game that has to be turned into a hypergraphical game named  $\tilde{G}$ .  $\tilde{G}$ ’s players are pairs  $(i, \theta_i)$ , their action sets still are  $A_i$ :

```

Definition HR_player : finType := [finType of {i : I & T i}].
Definition HR_action (i_ti : HR_player) : eqType := A (projT1 i_ti).

```

Strategy profiles of  $G$  and of  $\tilde{G}$  are in one-to-one correspondance. Every strategy profile  $\sigma : \text{iprofile } A \ T$  of  $G$ , that is,  $\sigma : \prod_{i \in I} (\Theta_i \rightarrow A_i)$ , is flattened to  $\tilde{\sigma} : \text{cprofile } (\text{fun } i\_ti : \{i : I \ \& \ T \ i\} \Rightarrow A \ (\text{val } i))$  in  $\tilde{G}$ , that is,  $\tilde{\sigma} : \prod_{(i, \theta_i) \in I \times \Theta_i} A_i$ . E.g. in a 2-player game with 2 types per player,  $\sigma = (\sigma_1, \sigma_2)$  is flattened to  $(\sigma_1(\theta_1), \sigma_1(\theta'_1), \sigma_2(\theta_2), \sigma_2(\theta'_2))$ . This “dependent uncurrying” is performed by the following function:

```

Definition flatten I (T : I -> finType) A (sigma : iprofile A T) :=
[ffun i_ti => sigma (projT1 i_ti) (projT2 i_ti)].

```

## 6.1 The Direct Transform

The direct transform only holds for Dempster’s conditioning, which is made on-the-fly on prior focal elements. It is suitable for any XEU. Starting from a Bel game  $G$ , we construct a local game  $e_B$  for each prior focal element  $B$ . Vertex  $(i, \theta_i)$  plays in  $B$  iff  $\theta_i$  is possible in  $B$ , that is, if  $\exists \theta' \in B, \theta_i = \theta'_i$ . Its local utility in  $e_B$  is the “part of XEU” computed over  $B'$ , the subset of  $B$  on which the mass shall be transferred during Dempster’s conditioning.

► **Definition 17** (Direct transform of a Bel game). *The direct transform of a Bel game  $G = (I, (A_i, \Theta_i, u_i)_{i \in I}, m)$  is the hypergraphical game  $\tilde{G} = (\tilde{I}, \tilde{E}, (\tilde{A}_{(i, \theta_i)})_{(i, \theta_i) \in \tilde{I}}, (\tilde{u}_{(i, \theta_i)}^e)_{e \in \tilde{E}, (i, \theta_i) \in e})$ :*

- $\tilde{I} = \{(i, \theta_i) \mid i \in I, \theta_i \in \Theta_i\}$ ,  $\tilde{E} = (e_B)_{B \subseteq S_m}$ ,  $e_B = \{(i, \theta_i) \mid \theta \in B, i \in I\}$ ,  $\tilde{A}_{(i, \theta_i)} = A_i$ ,
- for each  $e_B \in \tilde{E}$ ,  $(i, \theta_i) \in e_B$  and  $\tilde{\sigma} \in \tilde{A}$ , let us pose  $\tilde{v}_i^{\tilde{\sigma}}(\theta) = u_i(\tilde{\sigma}^\theta, \theta)$  in:
- $\tilde{u}_{(i, \theta_i)}^{e_B}(\tilde{\sigma}_{e_B}) = m(B) \times (\varphi^{\text{XEU}}(\tilde{v}_i^{\tilde{\sigma}})(B \cap \{\theta' \mid \theta'_i = \theta_i\})) / \text{Pl}(\{\theta' \mid \theta'_i = \theta_i\})$ .

Formally, let  $G$  be a proper Bel game w.r.t. Dempster’s conditioning and  $\text{fXEU}$  a  $\varphi$  function:

```

Variable (proper_G : proper_belgame G (Dempster_conditioning R Tn))
(fXEU : {ffun Tn -> R} -> {ffun {set Tn} -> R}).

```

Then, let  $\tilde{G}$ ’s local games be indexed by focal elements, i.e., sets of type configurations:

```

Definition HRdirect_localgame := [finType of {set Tn}].

```

A vertex  $(i, \theta_i)$  plays in the local game  $e_B$  iff  $\theta_i$  is possible in  $B$ :

```

649
650 Definition HRdirect_plays_in (lg : HRdirect_localgame) (i_ti : HR_player) : bool
651 := [∃ t : Tn, [∧ t \in lg & t (projT1 i_ti) == projT2 i_ti]].
652

```

Then, local utility functions are given by a function which constructs from a local profile  $p$  and a type configuration  $\theta$ , the cprofile  $(p_{(1,\theta_1)}, \dots, p_{(n,\theta_n)})$ . This function has type:

```

653
654 Definition HRdirect_mkprofile lg i_ti (Hi_ti : HRdirect_plays_in lg i_ti)
655 (p : HRdirect_localprof lg) (t : Tn) : profile.
656

```

Local utility in a local game  $e_B$  is the part of the XEU computed from the prior focal element  $B$ . Note that Dempster's conditioning transfers masses from  $B$  to  $B \cap \{\theta' \in \Theta \mid \theta'_i = \theta_i\} = B \cap (\text{event\_ti } \theta_i)$  so the local utility amounts to an on-the-fly Dempster's conditioning. The resulting HG game is finally built from local utility functions:

```

657
658 Definition HRdirect_u : ∀ lg, HRdirect_localprof lg → HRdirect_localplayer lg → R
659 := fun lg p x ⇒ let (i_ti, Hi_ti) := x in let (i, ti) := i_ti in
660   G.1 lg * fXEU [ffun t ⇒ G.2 (HRdirect_mkprofile Hi_ti p t) t i]
661   (lg &: (event_ti ti)) / Pl G.1 (event_ti ti).
662
663 Definition HRdirect : cgame R HR_action := hg_game HRdirect_u.
664

```

► **Theorem 12** (Correctness of the direct transform). *For any proper Bel game  $G$ , Player  $i$  with type  $\theta_i$ , XEU function  $\varphi^{\text{XEU}}$ , and profile  $\sigma$ , we have  $\text{XEU}(m(\cdot \mid_D \theta_i))(\varphi^{\text{XEU}}(u_{i,\sigma})) = \tilde{u}_{(i,\theta_i)}(\text{flatten}(\sigma))$ . Thence, Nash equilibria of  $G$  and  $\tilde{G}$  are in one-to-one correspondence:*

```

665
666 Theorem HRdirect_correct (i : I) (ti : T i) (p : iprofile A T) :
667   belgame_utility fXEU properG p ti = HRdirect (flatten p) (existT _ i ti).
668
669 Theorem HRdirect_eqNash (p : iprofile A T) :
670   BelG_Nash_equilibrium fXEU proper_G p ↔ Nash_equilibrium HRdirect (flatten p).
671

```

## 6.2 The Conditioned Transform

The conditioned transform holds for any conditioning and XEU. Starting from a Bel game  $G$ , all the conditioning “given  $\theta_i$ ” are pre-computed, let  $\mathcal{S}^*$  be the union of all posterior focal sets (i.e., the set of all possible focal elements given any  $\theta_i$ ). Each  $B \in \mathcal{S}^*$  leads to a local game. As in the direct transform, a vertex  $(i, \theta_i)$  plays in  $e_B$  if  $\theta_i$  is possible in  $B$ . Its utility in  $e_B$  is the part of XEU computed over the posterior focal element  $B$ . Note that  $(i, \theta_i)$ 's local utility in  $B$  may be 0, if  $B$  is not focal in the posterior “given  $\theta_i$ ”. Formally speaking:

► **Definition 18** (Conditioned transform). *The conditioned transform of a Bel game  $G = (I, (A_i, \Theta_i, u_i)_{i \in I}, m)$  is the hypergraphical game  $\tilde{G} = (\tilde{I}, \tilde{E}, (\tilde{A}_{(i,\theta_i)})_{(i,\theta_i) \in \tilde{I}}, (\tilde{u}_{(i,\theta_i)}^e)_{e \in \tilde{E}, (i,\theta_i) \in e})$ :*

- $\tilde{I} = \{(i, \theta_i) \mid i \in I, \theta_i \in \Theta_i\}$ ,  $\tilde{E} = (e_B)_{B \in \mathcal{S}^*}$ ,  $e_B = \{(i, \theta_i) \mid \theta \in B, i \in I\}$ ,  $\tilde{A}_{(i,\theta_i)} = A_i$ ,
- for each  $e_B \in \tilde{E}$ ,  $(i, \theta_i) \in e_B$  and  $\tilde{\sigma} \in \tilde{A}$ , let us pose  $\tilde{v}_i^{\tilde{\sigma}}(\theta) = u_i(\tilde{\sigma}^\theta, \theta)$  in:

$$\tilde{u}_{(i,\theta_i)}^{e_B}(\tilde{\sigma}_{e_B}) = m(B \mid \theta_i) \times f_{\tilde{v}_i^{\tilde{\sigma}}}^{\text{XEU}}(B).$$

Formally, let fXEU be any  $\varphi^{\text{XEU}}$ , cond any conditioning, and  $G$  be proper w.r.t. cond:

```

692
693 Variables (fXEU: (Tn → R) → {set Tn} → R)
694 (cond : conditioning R Tn) (proper_G : proper_belgame G cond).
695

```

After similar definitions for HRcond\_localgame and HRcond\_plays\_in, we define:

```

696
697 Definition HRcond_u : ∀ lg, HRcond_localprof lg → HRcond_localplayer lg → R
698 := fun lg p x ⇒ let (i_ti, Hi_ti) := x in let (i, ti) := i_ti in
699   let kn := cond G.1 (event_ti ti) (is_revisable proper_G ti) in
700   kn lg * fXEU [ffun t ⇒ G.2 (HRcond_mkprofile Hi_ti p t) t i] lg.
701
702 Definition HRcond : cgame R HR_action := hg_game HRcond_u.
703

```

704 ► **Theorem 13** (Correctness of the conditioned transform). *For any proper Bel game  $G$ , Player  $i$*   
 705 *with type  $\theta_i$ , conditioning  $c$ , XEU function  $\varphi^{\text{XEU}}$ , profile  $\sigma: \text{XEU}(m(\cdot | c \theta_i))(\varphi^{\text{XEU}}(u_{i,\sigma})) =$*   
 706  *$\tilde{u}_{(i,\theta_i)}(\text{flatten}(\sigma))$ . Thence, Nash equilibria of  $G$  and  $\tilde{G}$  are in one-to-one correspondence:*

```
707 Theorem HRcond_correct (i : I) (ti : T i) (p : iprofile A T) :
708   belgame_utility fXEU proper_G p ti = HRcond (flatten p) (existT _ i ti).
709 Theorem HRcond_eqNash (p : iprofile A T),
710   BelG_Nash_equilibrium fXEU proper_G p <=> Nash_equilibrium (HRcond) (flatten p).
711
712
```

### 713 6.3 The TBM Transform

714 The TBM transform is designed for the Transferable Belief Model [33], in which knowledge  
 715 is first revised using Dempster’s conditioning, then decision is eventually made w.r.t. a  
 716 probability distribution BetP which is deduced from the bpa  $m$  (Definition 14). Here we  
 717 benefit from BetP’s 1-additivity to produce a low-complexity hypergraph: local games  
 718 correspond to single states of the world, that is, they all involve only  $n$  players.

719 ► **Definition 19** (TBM transform). *Let  $G = (I, (A_i, \Theta_i, u_i)_{i \in I}, m)$  be a Bel game; it is TBM-*  
 720 *transformed into the hypergraphical game  $\tilde{G} = (\tilde{I}, \tilde{E}, (\tilde{A}_{(i,\theta_i)})_{(i,\theta_i) \in \tilde{I}}, (\tilde{u}_{(i,\theta_i)}^e)_{e \in \tilde{E}, (i,\theta_i) \in e})$  s.t.:*  
 721  $\tilde{I} = \{(i, \theta_i) \mid i \in I, \theta_i \in \Theta_i\}$ ,  $\tilde{E} = (\text{Players}(\theta))_{\theta \in \Theta}$ ,  $\tilde{A}_{(i,\theta_i)} = A_i$ ,  
 722  $\tilde{u}_{(i,\theta_i)}^e(\tilde{\sigma}_e) = \text{BetP}_{(i,\theta_i)}(\theta) \times u_i(\sigma^\theta, \theta)$ .

723 Formally, let  $\text{cond}$  be a conditioning and  $G$  be a proper Bel game w.r.t.  $\text{cond}$ ;  $\tilde{G}$ ’s local  
 724 games are indexed by type configurations, and  $(i, \theta_i)$  plays in  $e_{\theta'}$  iff  $\theta_i = \theta'$ :

```
725 Variables (cond : conditioning R Tn) (proper_G : proper_belgame cond).
726 Definition HRTBM_localgame : finType := Tn.
727 Definition HRTBM_plays_in : HRTBM_localgame -> pred HR_player :=
728   fun lg i_ti => lg (projT1 i_ti) == projT2 i_ti.
729
```

731 Local utilities are computed w.r.t. the “pignistic” distribution BetP:

```
732 Definition HRTBM_u : ∀ lg, HRTBM_localprof lg -> HRTBM_localplayer lg -> R :=
733   fun lg p x => let (i_ti, _) := x in let (i, ti) := i_ti in
734     let betp := BetP (cond G.1 (event_ti ti) (is_revisable proper_G ti)) in
735     dist betp lg * G.2 (HRTBM_mkprofile p) lg i.
736 Definition HRTBM : cgame R HR_action := hg_game HRTBM_u.
737
738
```

739 ► **Theorem 14** (Correctness of the TBM transform). *For any proper Bel game  $G$ , Player  $i$  with*  
 740 *type  $\theta_i$ , conditioning  $c$ , and profile  $\sigma$ , TBEU  $(m(\cdot | c \theta_i))(\varphi^{\text{TBEU}}(u_{i,\sigma})) = \tilde{u}_{(i,\theta_i)}(\text{flatten}(\sigma))$ .*  
 741 *Thence, Nash equilibria of  $G$  and  $\tilde{G}$  are in one-to-one correspondence:*

```
742 Theorem HRTBM_correct (i : I) (ti : T i) (p : iprofile A T) :
743   belgame_utility fTBEU proper_G p ti = HRTBM (flatten p) (existT _ i ti).
744 Theorem HRTBM_eqNash (p : iprofile A T),
745   BelG_Nash_equilibrium fTBEU proper_G p <=> Nash_equilibrium HRTBM (flatten p).
746
```

## 748 7 Conclusion and Perspectives

749 In this paper, a Coq/SSReflect formalization of Bel games has been presented. It provides a  
 750 generic class of games of incomplete information, built upon the Dempster-Shafer theory of  
 751 belief functions. This framework makes it possible to capture (lack of) knowledge better than  
 752 usual game models based on probability. Following Howson’s and Rosenthal’s approach, three  
 753 different algorithms transforming such incomplete games into standard complete-information  
 754 games have been formalized, and the fact that these transforms preserve equilibria (as long

755 as we comply with Table 4) has been formally verified. Thus, Bel games can ultimately be  
 756 solved using state-of-the-art, effective algorithms for complete games.

757 This work provides strong guaranties on the correctness of the transforms, so that  
 758 game theorists may rely on them without any concern about correctness. Furthermore, the  
 759 formalization allowed us to identify subtleties that were left implicit in the definitions (e.g.,  
 760 the conditioning precondition), as well as to help improving the proofs, both in their flow  
 761 and in their prose. Last, generic lemmas that proved useful during our formalization effort  
 762 have been proposed for integration in the MathComp library.

763 This work opens several research directions, both on the theoretical side and on the  
 764 formal verification side. On the one hand, we aim at extending this result with other  
 765 decision-theoretic approaches, e.g., partially-ordered utility aggregations for belief function  
 766 and other non-additive-measure approaches (Choquet capacities of order 2, RDU). On the  
 767 other hand, we would like to focus on complexity proofs which, albeit not safety-critical,  
 768 play a key role when choosing one transform over the other. Eventually, we would like to  
 769 encompass this work into a larger library of decision under uncertainty, fostering further  
 770 developments on related models and proofs.

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