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Games of Incomplete Information: a Framework Based on Belief Functions*

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Abstract

This paper proposes a model for incomplete games where the knowledge of the players is represented by a Dempster-Shafer belief function. Beyond an extension of the classical definitions, it shows such a game can be transformed into an equivalent hypergraphical complete game (without uncertainty), thus generalizing Howson and Rosenthal's theorem to the framework of belief functions and to any number of players. The complexity of this transformation is finally studied and shown to be polynomial in the degree of k -additivity of the mass function.

Keywords: Game theory, Incomplete games, Belief functions, Choquet integrals

1. Introduction

Game theory [1, 2] proposes a powerful framework to capture decision problems involving several agents. In non-cooperative games of complete information, the players do not coordinate their actions but each of them knows every-
5 thing about the game: the players, their available actions and all their utilities. This assumption of complete knowledge cannot always be satisfied. In the real world indeed, players are not so well informed and have only limited knowledge

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about the game. This is why Bayesian games of incomplete information have been proposed [3]. Nevertheless, the Bayesian hypothesis is strong, and requires
10 a good knowledge of the environment. For instance, in case of ignorance, the Bayesian way amounts to assuming equiprobability, but this can lead to a model that does not fit with the agents' behavior (e.g. see Ellsberg's paradox [4]).

In the present paper, we propose a new kind of game of incomplete information, which we call *Bel game*. Agents have a partial knowledge, represented
15 by a Dempster-Shafer belief function [5, 6], and cardinal utilities, but do not necessarily make the equiprobability assumption. The underlying decision rule is generally the Choquet integral based on the *Bel* measure [7], which amounts to the maximization of the worst expected utility [8, 9]. Bel games as defined here are also compatible with the transferable belief model [10], which amounts
20 to extracting the pignistic probability when the decision is to be made, and with Jaffray's linear utility [11].

We then follow the line defined by Howson and Rosenthal [12] who have shown that any 2-player Bayesian game can be transformed into a complete knowledge *polymatrix game* [13]. In this paper, we show that such a transfor-
25 mation is possible for Bel games, and for any number of agents, producing a *hypergraphical game* [14]. A notable consequence of this result is that the algorithmics developed for hypergraphical games [15, 16] can be reused for the search of Nash equilibria in Bel games.

2. Background and motivations

30 To illustrate and motivate our work, we will use the following example inspired by the *murder of Mr. Jones* [10], where the suspects are *Peter, Paul and Mary*.

Example 1 (Peter, Quentin and Rose). *Two agents, named Agent 1 and Agent 2, are independently looking for a business association, with either Peter
35 (P), Quentin (Q), or Rose (R). The point is that a crime has been committed, for which these three people are suspected. Several testimonies, not very reliable,*

allow to estimate that there is 50% of chance that the culprit is a man (P or Q), and 50% of chance that it is a woman (R).

As to the interest of the associations, making the deal with an innocent leads to a
 40 payoff of \$6k (to be shared between the people making the deal), while associating
 with a guilty person produces no payoff (\$0k). Moreover, Agent 1 is investigating
 about P and will eventually knows whether he is guilty or not. Similarly, Agent 2
 will knows whether R is guilty before making the decision.

The Bayesian approach is not relevant here. Indeed, if Agent 1 learns that
 45 P is innocent the probability of guilt should become 1/2 for Q and 1/2 for
 R . However, in a purely Bayesian view equiprobability would be applied and
 the prior probability of guilt would be 1/4 to P and 1/4 to Q . Then, after
 conditioning, Agent 1 would get a probability of 1/3 to Q and 2/3 to R .

2.1. Dempster-Shafer's theory of evidence

50 Let us first look at the epistemic aspect of the problem. The prior knowl-
 edge is simply that $P(\{P, Q\}) = P(\{M\}) = \frac{1}{2}$, and nothing more. The kind
 of knowledge at work here is well captured in Dempster-Shafer's theory of ev-
 idence, that does not restrict probability assignments to elements of the frame
 of discernment:

55 **Definition 1 (Mass function).** *A mass function for a frame of discernment*
 Ω (or “bpa” for basic probability assignment) *is a function* $m : 2^\Omega \rightarrow [0, 1]$ *such*
that $m(\emptyset) = 0$ *and* $\sum_{A \subseteq \Omega} m(A) = 1$.

A set with a nonzero mass is called a *focal element* and the set of focal
 elements is denoted \mathcal{S}_m . Two dual measures on 2^Ω derive from m :

$$\text{Bel}(A) = \sum_{B \in \mathcal{S}_m, B \subseteq A} m(B) \quad \text{and} \quad \text{Pl}(A) = \sum_{B \in \mathcal{S}_m, B \cap A \neq \emptyset} m(B).$$

$\text{Bel}(A)$ (resp. $\text{Pl}(A)$) estimates to what extent A is implied by (resp. is com-
 patible with) the knowledge captured by m .

60 Probabilities are the special case of belief functions where mass functions (called distributions) are 1-additive: the focal elements are then singletons. k -additivity is more generally defined as follows:

Definition 2 (k -additivity). *A mass function is k -additive if all its focal elements are at most of size k , i.e., $\forall B \in \mathcal{S}_m, |B| \leq k$.*

65 Probability theory is recovered as a special case when the available information is uncertain, but precise. When there is on the contrary only one focal set B , i.e., when the belief function describes a piece of evidence that tell us that ω is in B for sure, and nothing more, the information is certain but imprecise. Following this interpretation, the mass function is seen as a generalized set [17].

70 Belief function can alternatively be understood as a particular case of imprecise probability theory. A belief function Bel and its dual Pl indeed delimit the lower and upper bounds of a probability family $\mathcal{F} = \{\text{Pr} \mid \forall A, \text{Bel}(A) \leq \text{Pr}(A) \leq \text{Pl}(A)\}$ - \mathcal{F} is called the credal set of m . It is clear that this set is convex, i.e., that any element of \mathcal{F} can be obtained by distributing each mass
75 $m(B)$ among the elements of B . Of course, not any lower probability measure is a belief functions [18].

Now, the two views of belief functions belong to different theoretical frameworks [19, 20]. In particular, the two theories have different rules of conditioning. In a pure DS theory, in the conditioning of m by C , the mass assigned to a focal
80 set B is transferred to their non-empty intersection. The conditioning at work here is Dempster's rule [5] (see [21] for more details):

Definition 3 (Dempster conditioning). *For any nonempty $A, C \subseteq \Omega$, with $\text{Pl}(C) > 0$ (at least one focal element intersects C),*

$$m|_C^{\text{Dem}}(A) := K_C \cdot \sum_{\substack{B \in \mathcal{S}_m \\ C \cap B = A}} m(B),$$

where $K_C = 1/\text{Pl}(C)$ is a normalization factor, constant for a given subset $C \subseteq \Omega$.

Masses of $m|_C^{\text{Dem}}$ can be computed using a simple algorithm. The mass $m(B)$
85 of any focal element $B \in \mathcal{S}_m$ is transferred to the subset $B \cap C$ if it is nonempty,
and discarded otherwise. Thus all masses of $m|_C^{\text{Dem}}$ can be computed through a
single loop over \mathcal{S}_m , followed by a normalization, in time linear w.r.t. the size
of m .

Dempster conditioning preserves the size and the k -additivity of m :

$$|\mathcal{S}_m| \geq |\mathcal{S}_{m|_C^{\text{Dem}}}| \quad \text{and} \quad m \text{ is } k\text{-additive} \implies m|_C^{\text{Dem}} \text{ is } k\text{-additive}$$

The Fagin-Halpern conditioning [22] on the contrary derives from the inter-
90 pretation of belief functions in the theory of imprecise probabilities.

Definition 4 (Fagin-Halpern conditioning). *For any non-empty C with $\text{Bel}(C) > 0$ (i.e. at least one focal element is included in C),*

$$\text{Bel}(A|C) = \inf_{\Pr \in \mathcal{F}_m} \Pr(A|C) = \frac{\text{Bel}(A \cap C)}{\text{Bel}(A \cap C) + \text{Pl}(\bar{A} \cap C)}$$

[22] have shown that the above-defined measure is a Bel measure. An algo-
rithm is proposed in [20] which allows the computation of the bpa $m|_C^{\text{FH}}$ corre-
95 sponding to $\text{Bel}(\cdot|C)$. This algorithm, which relies on the Moebius transform,
is not polynomial w.r.t. the size of m : it requires $O(k \cdot s^3 \cdot 2^s \cdot s!)$ operations
where s is the number of focal elements of m and k its degree of additivity.
The number of focal elements of $m|_C^{\text{FH}}$ may be much greater than the number
of focal elements of the original Bel and the FH conditioning doesn't preserves
100 nor the size, neither the k -additivity of the mass function².

2.2. Decision making with belief functions

Let us now consider belief functions in a (mono-agent) decision making con-
text. Following Savage's modelling of decision making under uncertainty [23],

²Consider for instance a frame of discernment $\Omega = \{\omega_1, \dots, \omega_m\}$ and a 2-additive mass function m such as $m(\{\omega_i\}) > 0$ for all i and $m(\{\omega_i, \omega_j\}) > 0$ for all $i \neq j$. Then, for any nonempty $C \subset \Omega$, each subset of $B \subseteq C$ is a focal element of $\text{Bel}(\cdot|C)$ – thus $|\mathcal{S}_{m|_C^{\text{FH}}}| = 2^{|C|}$ and $\text{Bel}(\cdot|C)$ is $|C|$ -additive.

a decision (or “action”) is a function $a : \Omega \rightarrow X$ where Ω is the set of possible
 105 states, as previously, and X is the set of possible outcomes. The preferences of
 an agent are represented by an utility function $u : X \rightarrow \mathbb{R}$.

The multi prior model and the discrete Choquet integral

A belief function Bel and its dual Pl delimit the lower and upper bounds of a
 probability family $\mathcal{F} = \{Pr \mid \forall A, \text{Bel}(A) \leq \text{Pr}(A) \leq \text{Pl}(A)\}$, and thus, for each
 110 decision, a full range of values for its expected utility. Gilboa and Schmeidler
 [9, 8] consider a pessimistic agent and propose to evaluate each decision by the
 minimum of the possible expected utilities.

Definition 5 (MPEU). $\text{MPEU}(a) = \min_{Pr \in \mathcal{F}} (\sum_{\omega} \text{Pr}(\omega) \cdot u(a(\omega)))$

Another line of thought is to consider that belief functions are particular
 115 capacity measures, and to compute the global merit of an act on the basis of
 its Choquet value (for a theoretical justification of the use of Choquet integral
 in decision making under (non-probabilistic) uncertainty, see [24, 25]). When
 the capacity is a belief function, the Choquet expectation writes:

Definition 6 (Choquet expected utility (CEU)). Let $\Lambda(a) = \{\lambda_1 \leq \dots \leq$
 $\lambda_{|\Lambda(a)|}\}$ be the set of utility values reached by an action a , labelled by increasing
 order, and $E_{\lambda_i}(a) = \{\omega \mid u(a(\omega)) \geq \lambda_i\}$ denote the set of worlds for which the
 utility of action a is at least λ_i . The Choquet Expected Utility of a is :

$$\text{CEU}(a) = \lambda_1 + \sum_{i=2}^{|\Lambda(a)|} (\lambda_i - \lambda_{i-1}) \times \text{Bel}(E_{\lambda_i}(a)).$$

The CEU has a simple expression in terms of the mass function:

$$\text{CEU}(a) = \sum_{B \in \mathcal{S}_m} m(B) \times \min_{\omega \in B} u(a(\omega)).$$

[26] have shown that the MPEU value of a decision is equal to its CEU value
 120 with respect to the belief functions.

We recover here the double interpretation of belief functions, in terms of
 generalized set or in terms of imprecise probability. It should nevertheless be

recalled that they do differ, especially in a dynamic context, where a conditioning must be applied.

125 *Jaffray's expected utility (JEU)*

Jaffray's [11] expected utility (JEU) is defined directly in terms of the mass function, and generalizes the CEU by allowing one to modulate the agent's pessimism locally, using a series of Hurwicz coefficients.

$$\text{JEU}(a) = \sum_{B \in \mathcal{S}_m} m(B) \times (\alpha_{B_*(a), B^*(a)} \times B_*(a) + (1 - \alpha_{B_*(a), B^*(a)}) \times B^*(a))$$

where $B_*(a) = \min_{\omega \in B} u(a(\omega))$ and $B^*(a) = \max_{\omega \in B} u(a(\omega))$.

130 The α_{x_i, x_j} coefficients represent the agent's pessimism and must be elicited for any pair (x_i, x_j) with $i < j$; there is a quadratic number of such pairs (w.r.t. the possible utility values). Note that $\text{CEU} = \text{JEU}$ if all coefficients α_{x_i, x_j} are equal to 1.

The transferable belief model (TBEU)

135 Smets and Kennes' model [10] proceeds in two steps: at the credal level, knowledge is represented by a belief function and revised by Dempster's rule of conditioning; at the pignistic level (when a decision takes place), a probability law p is constructed: $p(\omega) = \sum_{B \in \mathcal{S}_m} m(B)/|B|$. The agent then maximizes his expected utility with regards to p .

140 *2.3. Game theory*

A simultaneous game of complete information models a situation where each agent makes a decision (the term "action" is rather used in game theory) without coordination with the other ones – the final utility of each agent depending on the actions chosen by all agents.

145 **Definition 7 (Complete game).** *A simultaneous game of complete information (also called complete game) is a tuple $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$ where:*

- $N = \{1, \dots, n\}$ is a finite set of agents (or “players”),
- A_i is the set of actions of Agent i ; the set $A := \prod_{i \in N} A_i$ contains all the possible combinations of actions a.k.a. “profiles”,
- $u_i : A \rightarrow \mathbb{R}$ is the utility function of Agent i .

A mixed strategy for player i is a probability distribution on A_i . The strategy is said to be pure when only one action receives a non-zero probability.

A pure (resp. mixed) strategy profile is a vector $p = (p_1, \dots, p_n)$ which specifies a pure (resp. mixed) strategy for each player.

In the following, we will use the following notations: for any vector $v = (v_1, \dots, v_n)$ in some product domain $V = \prod_{i \in N} V_i$ and for any $e \subseteq N$, v_e is the restriction of v to e and $V_e = \prod_{i \in e} V_i$. By abuse of notation, we write v_i for $v_{\{i\}}$. For any i , $-i$ denotes the set $N \setminus \{i\}$, i.e. $v_{-i} = (v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n) \in V_{-i} = \prod_{j \neq i} V_j$. Thus, v_{-i} is the restriction of v to all players but i . Finally, “.” denotes the concatenation, e.g., $v'_i.v_{-i} = (v_1, \dots, v_{i-1}, v'_i, v_{i+1}, \dots, v_n)$. Hence $a = a_i.a_{-i}$ belongs to A and given two profiles $a, a' \in A$, $a'_i.a_{-i}$ denotes the profile a where a_i is replaced with a'_i .

Because the strategies can be randomized, the global utility for a player of a mixed strategy profile p is defined as the expected utility (EU) of u_i according to the probability distribution it induces over A (obviously, when the strategy is pure, EU_i is equal to the utility value given by u_i):

Definition 8 (Utility of a strategy). Given a strategy profile p in a complete game $(N, (A_i)_{i \in N}, (u_i)_{i \in N})$, the expected utility of player i is defined by:

$$\text{EU}_i(p) = \sum_{a \in A} \left(\prod_{j \in N} p_j(a_j) \right) \times u_i(a).$$

Nash equilibria are the profiles in which no player can unilaterally increase his/her utility by changing his/her own strategy.

Definition 9 (Nash equilibrium [2]). A strategy profile p is a Nash equilibrium iff for any $i \in N$, there exists no p'_i such that $\text{EU}_i(p'_i, p_{-i}) > \text{EU}_i(p)$.

When the utility functions are described explicitly, G is said to be in standard normal form (SNF). SNF representations become spatially costly when the number of players increases ($O(n\alpha^n)$ for a game with n players and α actions per player). More succinct forms have been proposed, that suit cases where utility functions can be decomposed as a sum of smaller utility functions, namely hypergraphical games [14] and their particular cases, polymatrix games [13] and graphical games [27].

Definition 10 (Hypergraphical game).

A hypergraphical game is a tuple $G = (N, E, (A_i)_{i \in N}, (u_i^e)_{e \in E, i \in e})$ where:

- N is a set of players,
- $E = \{e_1, \dots, e_m\}$ is a multiset of subsets of N ((N, E) is an hypergraph)
- For each $e \in E$, $(e, (A_i)_{i \in e}, (u_i^e)_{i \in e})$ is a complete game.

The global utility of each Agent i sums the local utilities: $u_i(a) = \sum_{e \in E} u_i^e(a_e)$.

Polymatrix games are hypergraphical games with 2-players local games:

$$\forall e \in E, |e| = 2.$$

This framework assumes that each player knows everything about the game: the players, the actions available to each player, all their utilities for each combination of actions, etc. The assumption of complete knowledge cannot always be satisfied. In the real world indeed, players have only a limited knowledge about the outcomes of their strategies – the final outcomes may depend on an ill-known event (in our example 1, the payoff for making the deal with one of P , Q , or R depends on whether they are guilty or innocent).

Harsanyi [3] proposed games of incomplete information as a way to capture such situations (see also [28], for more details). A game of incomplete information can be first understood as a set of possible classical games (of complete information) – one for each possible world $\omega \in \Omega$. Players don't know exactly which world is the real one, but may have some knowledge about it. Just before playing, each player i will receive some information $\tau_i(\omega^*)$ about the real

200 world ω^* . τ_i maps any world to an element θ_i of a set Θ_i called the set of
“types” of Agent i . After having observed $\tau_i(\omega^*)$, Agent i knows more about
the real game, but several games may still be plausible. The player then con-
ditions his/her knowledge on $\tau_i(\omega^*)$ and decides which action to play. Notice
that the different agents may receive different pieces of information and thus
205 have a different posterior knowledge. The question is then, for each player, to
determine a strategy (either an action, or a probabilistic strategy) for each of
his/her possible types.

Harsanyi has shown that such games can be described on the space of types
 $\Theta = \Theta_1 \times \dots \times \Theta_n$ (the underlying worlds are omitted). The idea of Harsanyi
210 when defining types is that this concept can encapsulate every piece of informa-
tion agents may have access to. It includes the agent-observable world status,
but also their beliefs on other agents and their introspective mental states.

3. Bel games

Bayesian games are games of incomplete information where prior knowledge
215 is captured by a probability measure. To capture problems where the Bayesian
assumption is not obeyed (as in our motivating example), we propose the more
general framework of Bel games:

3.1. Definitions

Definition 11 (Bel game). *A Bel game G is defined as a tuple*
220 *$(N, (A_i)_{i \in N}, (\Theta_i)_{i \in N}, (u_i)_{i \in N}, m)$ where:*

- $N = \{1, \dots, n\}$ is a finite set of players,
- A_i is the set of actions of player i ; $A = \prod_{i \in N} A_i$ denotes the set of all
action profiles,
- Θ_i is the set of types of player i ; $\Theta = \prod_{i \in N} \Theta_i$ denotes the set of all type
225 configurations,

- $m : 2^\Theta \rightarrow [0, 1]$ is the mass function describing the common prior knowledge,
- $u_i : A \times \Theta \rightarrow \mathbb{R}$ is the utility function of Agent i .

G is said to be in standard normal form iff the utility functions u_i and the
230 mass function m are given in extenso.

Bel games generalize Bayesian games, which are recovered when m is a probability distribution.

Example 2. Our running example (see Example 1) is captured by the Bel game $G = (N, (A_i)_{i \in N}(\Theta_i)_{i \in N}, (u_i)_{i \in N}, m)$ where:

- $N = \{1, 2\}$;
- $A_1 = \{P_1, Q_1, R_1\}$, $A_2 = \{P_2, Q_2, R_2\}$ (each agent chooses an associate).
- $\Theta_1 = \{\bar{P}, \bar{P}\}$, $\Theta_2 = \{\bar{R}, \bar{R}\}$ (Agent 1 investigates on Peter, Agent 2 investigates on Rose).
- $m : 2^\Theta \rightarrow [0, 1]$ has two focal elements: $m(\{(\bar{P}, \bar{R})\}) = 1/2$ (the murdered is a woman, thus necessarily Rose – in this case Agent 1 will learn \bar{P} and Agent 2 will learn \bar{R}) and $m(\{(P, \bar{R}), (\bar{P}, \bar{R})\}) = 1/2$ (the murderer is a man: Agent 2 necessarily learns \bar{R} but Agent 1 can learn either \bar{P} – which happens when Quentin is the murdered – or P – Peter is the murderer).
- Making a deal with a murderer has a utility value of 0, making a deal with an innocent leads to a utility of $\frac{6}{2} = 3$, unless the other agent approaches the same associate, in which case each agent receives $\frac{6}{3} = 2$. The utility functions are summarized below (Table 1). Null values (in gray) are given for the case where $\theta = (P, R)$ (both R and P are guilty) which is not a possible world.

250 Following Harsanyi’s approach of incomplete games, we consider the “ex interim” setting where each player plans a strategy for each of the types he/she

		$\theta_2 = \bar{\mathbf{R}}$			$\theta_2 = \mathbf{R}$				
			P_2	Q_2	R_2		P_2	Q_2	R_2
$\theta_1 = \mathbf{P}$	P_1	(0, 0)	(0, 3)	(0, 3)	P_1	(0, 0)	(0, 0)	(0, 0)	
	Q_1	(3, 0)	(2, 2)	(3, 3)	Q_1	(0, 0)	(0, 0)	(0, 0)	
	R_1	(3, 0)	(3, 3)	(2, 2)	R_1	(0, 0)	(0, 0)	(0, 0)	
			P_2	Q_2	R_2		P_2	Q_2	R_2
$\theta_1 = \bar{\mathbf{P}}$	P_1	(2, 2)	(3, 0)	(3, 3)	P_1	(2, 2)	(3, 3)	(3, 0)	
	Q_1	(0, 3)	(0, 0)	(0, 3)	Q_1	(3, 3)	(2, 2)	(3, 0)	
	R_1	(3, 3)	(3, 0)	(2, 2)	R_1	(0, 3)	(0, 3)	(0, 0)	

Table 1: Example 1: Utility matrices for each configuration of the types

can receive, ideally a strategy which is a best response to that of the other players. We thus adopt the definition of strategy proposed by Harsanyi's in the general context of incomplete games:

255 **Definition 12 (Pure and mixed strategies [3]).** *A pure (resp. mixed) strategy for player i in a Bel game is a function ρ_i which maps each "type" $\theta_i \in \Theta_i$ to an action of (resp. a probability over) A_i .*

A pure (resp. mixed) strategy profile is a vector $p = (p_1, \dots, p_n)$ which specifies a pure (resp. mixed) strategy for each player.

260 $\rho(\theta) = (\rho_1(\theta_1), \dots, \rho_n(\theta_n))$ denotes the profile which will be played if the configuration of types is θ .

The set of all pure strategy profiles is denoted $\Sigma = \prod_{i \in N} (\Theta_i \rightarrow A_i)$.

In the *ex interim* approach of incomplete games, when receiving her type θ_i , Agent i revises her knowledge – in a Bel game, her posterior knowledge over the joint type configuration is $m_{|\theta_i}$.

Let us first consider the case where the agents maximize their Choquet utility (this approach being compatible with both the evidential and the credal interpretation of belief functions). In this case the utility of a pure strategy

profile for Agent i of type θ_i , shall thus be defined as:

270 **Definition 13 (Choquet Expected Utility of a pure strategy profile).**

The utility of a pure strategy profile $\rho = (\rho_1, \dots, \rho_n)$, for Agent i of type θ_i , is:

$$\text{CEU}_{(i, \theta_i)}(\rho) = \sum_{B \in S_{m|_{\theta_i}}} m|_{\theta_i}(B) \times \min_{\theta' \in B, \theta'_i = \theta_i} u_i(\rho(\theta'), \theta').$$

where $m|_{\theta_i}$ denotes the conditioning compatible with the interpretation of the belief function (e.g. Demspster's conditioning or Fagin Halpern's)

275 Let us now consider mixed strategies. A mixed strategy profile ρ defines a probability distribution $\text{Pr}^\rho(\sigma) = \prod_{i \in N} \prod_{\theta_i \in \Theta_i} \rho_i(\theta_i)(\sigma_i(\theta_i))$ over the set Σ of pure strategy profiles. If we now merge Pr^ρ with m , we get a bpa m^ρ over $A \times \Theta$: to any element $X = \{(a, \theta), (a', \theta'), \dots\} \subseteq A \times \Theta$ correspond both a set of type configurations $B_X := \{\theta \mid (a, \theta) \in X\} \subseteq \Theta$ and a set of compatible pure strategy profiles $S_X := \{\sigma \mid \forall (a, \theta) \in X, \sigma(\theta) = a\} \subseteq \Sigma$. The mass of X is
 280 $m^\rho(X) = m(B_X) \times \sum_{\sigma \in S_X} \text{Pr}^\rho(\sigma)$, that is, X is focal if B_X is focal and some compatible pure strategy profiles are possible. Finally, Agent i receiving type θ_i conditions her knowledge which becomes $m|_{\theta_i}^\rho$. Hence the following definition of the utility of a mixed strategy profile:

Definition 14 (Choquet Expected Utility of a mixed strategy profile).

The utility of a mixed strategy profile $\rho = (\rho_1, \dots, \rho_n)$, for player i of type θ_i , is: $\text{CEU}_{(i, \theta_i)}(\rho)$

$$= \sum_{\sigma \in \Sigma} \left(\prod_{i \in N} \prod_{\theta_i \in \Theta_i} \rho_i(\theta_i)(\sigma_i(\theta_i)) \right) \times \sum_{B \in S_{m|_{\theta_i}^\rho}} m|_{\theta_i}^{\text{Dem}}(B) \times \min_{\theta' \in B} u_i(\sigma(\theta'), \theta')$$

285 It can be checked that Definition 14 amounts to Definition 13 when ρ is a pure strategy profile.

Now, recall that a strategy is a Nash equilibrium if no player can improve unilaterally his/her utility. This concept straightforwardly extends to Bel games:

Definition 15 (Nash equilibrium). *A mixed (resp. pure) strategy profile ρ*

290 *is a Nash equilibrium for CEU iff, whatever (i, θ_i) , there exists no mixed (resp. pure) strategy ρ'_i such that $\text{CEU}_{(i, \theta_i)}(\rho'_i, \rho_{-i}) > \text{CEU}_{(i, \theta_i)}(\rho)$.*

Example 3. Let ρ be the pure strategy where Agent 1 makes the deal with R when learning that P is guilty and with P otherwise, and Agent 2 joins Q when learning that R is guilty and R otherwise:

$$\rho_1(P) = R_1, \rho_1(\bar{P}) = P_1, \rho_2(R) = Q_2 \text{ and } \rho_2(\bar{R}) = R_2.$$

As usual with this “Peter Paul and Mary” example, the Demspter rule of conditionning is used.

- Consider Agent 1 receiving type P : the conditioned bpa, m_P^{Dem} , has only one focal element $\{(P, \bar{R})\}$, $K_P = 1/\frac{1}{2}$ and $m_P^{Dem}(\{(P, \bar{R})\}) = 1$. In short, Agent 1 knows that P is guilty and R is not. In the only possible configuration (P, \bar{R}) , ρ prescribes $\rho_1(P) = R_1$ for Agent 1 and $\rho_2(\bar{R}) = R_2$ for Agent 2. Then,

$$CEU_{(1,P)}(\rho) = m_P^{Dem}(\{(P, \bar{R})\}) \times u_1((R_1, R_2), (P, \bar{R})) = 1 \times 2 = 2.$$

- Consider now Agent 1 receiving \bar{P} : his/her revised knowledge, $m_{\bar{P}}^{Dem}$, has two focal elements, $\{(\bar{P}, R)\}$ and $\{(\bar{P}, \bar{R})\}$ (each with probability $\frac{1}{2}$, thus $K_{\bar{P}} = 1$). The strategy prescribes $\rho(\bar{P}) = P_1$ for Agent 1, who doesn't know whether Agent 2 learns R (and plays $\rho(R) = Q_2$) or \bar{R} (and plays $\rho(\bar{R}) = R_2$). Hence

$$CEU_{(1,\bar{P})}(\rho) = \frac{1}{2} \times u_1((P_1, R_2), (\bar{P}, R)) + \frac{1}{2} \times u_1((P_1, Q_2), (\bar{P}, R)) = 3$$

- Similarly, the bpa of Agent 2 receiving R , m_R^{Dem} , has only one focal element, $\{(\bar{P}, R)\}$ (thus $K_R = 1/\frac{1}{2}$) in which ρ prescribes P_1 for Agent 1 and Q_2 for Agent 2. Then

$$CEU_{(2,R)}(\rho) = 1 \times u_2((P_1, R_2), (\bar{P}, R)) = 1 \times 3 = 3.$$

- Finally, the bpa of Agent 2 receiving \bar{R} , $m_{\bar{R}}^{Dem}$, has one focal element, $\{(\bar{P}, \bar{R}), (P, \bar{R})\}$ and $K_{\bar{R}} = 1/\frac{1}{2}$. Agent 2 does not know whether Agent 1 receives \bar{P} or P . Since ρ prescribes Agent 1 to play P_1 in the first case, R_1 in the second one and prescribes Agent 2 to play R_2 in both cases,

$$\begin{aligned} CEU_{(2,\bar{R})}(\rho) &= 1 \times \min [u_2((P_1, R_2), (\bar{P}, \bar{R})), u_2((R_1, R_2), (P, \bar{R}))] \\ &= 1 \times \min(3, 2) = 2. \end{aligned}$$

320 *In this strategy, Agent 1 does not give the best possible response to Agent 2's strategy: when learning that P is guilty, he/she plays R_1 while knowing that in this case Agent 2 learns \bar{R} and thus plays R_2 . Let Agent 1 modify his/her strategy and play Q_1 when learning P – hence the strategy ρ' :*

$$\rho'_1(P) = Q_1, \rho'_1(\bar{P}) = P_1, \rho'_2(R) = Q_2, \rho'_2(\bar{R}) = R_2.$$

- 325
- $\text{CEU}_{(1,P)}(\rho') = K_P \times u_1((Q_1, R_2), (P, \bar{R})) = 1 \times 3 = 3,$
 - $\text{CEU}_{(1,\bar{P})}(\rho') = K_{\bar{P}} \times u_1((P_1, R_2), (\bar{P}, \bar{R})) + K_{\bar{P}} \times u_1((P_1, Q_2), (\bar{P}, R)) = 3,$
 - $\text{CEU}_{(2,R)}(\rho') = K_R \times u_2((P_1, Q_2), (\bar{P}, R)) = 1 \times 3 = 3,$
 - $\text{CEU}_{(2,\bar{R})}(\rho') = K_{\bar{R}} \times \min \left(u_2((P_1, R_2), (\bar{P}, \bar{R})), u_2((Q_1, R_2), (P, \bar{R})) \right) = 3.$

330 *It can be checked that with ρ' , each player gets his/her maximal possible utility (\$3k) – no player has incentive to deviate: ρ' is a pure Nash equilibrium.*

3.2. Credal Games and Evidential games

Bel games as defined above can be understood under the DS theory or under the theory of imprecise probabilities. Because Choquet expected utility is compatible with both theories, we have first considered the pessimistic Choquet
335 integral as a way to evaluate the utility of the agents. Let us briefly investigate the model in each of the two interpretations, with respect to the different ways of conditioning and to the different decision rules.

3.2.1. CEU, JEU and Pignistic Games in the DS theory of evidence

Let us first consider problems having an interpretation in the DS theory and
340 are thus based on the Dempster rule of conditioning - we call these games “Evidential games”. Several decision rules can be used in this context, namely the Choquet integral (CEU) used in the previous section, Jaffray’s linear utility (JEU) and the transferable belief model (TBEU). Let us capture all of them as particular cases of a generalized expected utility:

$$\text{XEU}(a) = \sum_{B \in \mathcal{S}_m} m(B) \times f_{u \circ a}^{\text{XEU}}(B)$$

345 If m is a probability distribution, then $EU(a) = XEU(a)$ in all three cases.

We find back the CEU, JEU and TBEU with:

- $f_{u \circ a}^{CEU}(B) = \min_{\omega \in B} u(a(\omega))$
- $f_{u \circ a}^{JEU}(B) = \alpha_B \min_{\omega \in B} u(a(\omega)) + (1 - \alpha_B) \max_{\omega \in B} u(a(\omega))$
- $f_{u \circ a}^{TBEU}(B) = \sum_{\omega \in B} \frac{u(a(\omega))}{|B|}$

350 As to CEU (resp. JEU), the proof is trivial: one has just to rewrite $f_{u \circ a}^{CEU}$ (resp. $f_{u \circ a}^{JEU}$) in the equation to get back the on-focal-set expression.

As to TBEU, we need to go back to the expected utility model using the distribution: $\text{BetP}_m(\omega) = \sum_{\substack{B \subseteq \mathcal{S}_m \\ \omega \in B}} \frac{m(B)}{|B|}$ as defined in [10], that is:

$$\begin{aligned}
 \sum_{\omega \in \Omega} \text{BetP}_m(\omega) \times u(a(\omega)) &= \sum_{\omega \in \Omega} \left(\sum_{\substack{B \subseteq \mathcal{S}_m \\ \omega \in B}} \frac{m(B)}{|B|} \right) \times u(a(\omega)) \\
 &= \sum_{B \in \mathcal{S}_m} \sum_{\omega \in B} \frac{m(B)}{|B|} \times u(a(\omega)) \\
 &= \sum_{B \in \mathcal{S}_m} m(B) \times \sum_{\omega \in B} \frac{u(a(\omega))}{|B|} \\
 &= \sum_{B \in \mathcal{S}_m} m(B) \times f_{u \circ a}^{TBEU}(B)
 \end{aligned}$$

Definition 16. *The utility of a mixed strategy profile $\rho = (\rho_1, \dots, \rho_n)$, for player i of type θ_i , is:*

$$XEU_{(i, \theta_i)}(\rho) = \sum_{\sigma \in \Sigma} \left(\prod_{i \in N} \prod_{\theta_i \in \Theta_i} \rho_i(\theta_i)(\sigma_i(\theta_i)) \right) \times \sum_{B \in \mathcal{S}_{m|_{\theta_i}}^{Dem}} m_{|\theta_i}^{Dem}(B) \times f_{v_i^\sigma}^{XEU}(B)$$

where the Dempster conditioning is used and $v_i^\sigma(\theta) = u_i(\sigma(\theta), \theta)$

Using $XEU = CEU$, we find back Definition 14.

355 The definitions of pure and mixed Nash equilibria remain unchanged, i.e. ρ is a pure (resp. mixed) Nash equilibrium for XEU iff, whatever (i, θ_i) , there exists no pure (resp. mixed) strategy ρ'_i such that $XEU_{(i, \theta_i)}(\rho'_i, \rho_{-i}) > XEU_{(i, \theta_i)}(\rho)$.

3.2.2. Bel games in the credal interpretation

In the credal interpretation, the mass distribution actually defines a family of probability over the combinations of types, hence the use of the Fagin-Halpern conditioning. This interpretation is compatible with the Choquet-based decision rule (when the capacity used is a Bel measure, the Choquet value of a decision is equal to the minimum value of the expected utilities provided by the different probabilities of the family). [29] shows that it is also compatible with Jaffray's linear utility (JEU).

Notice that alternative rules of conditioning exist in this interpretation, the Strong conditioning [6, 17, 30], also called “geometrical conditioning”, which amounts to applying Jeffrey's rule when learning the categorical mass function m' such as $m'(C) = 1$ and the weak conditioning [31], seldom used because leading to strange results (for example $\text{Bel}^{\text{Weak}}(C | C) = \text{Bel}(C) / \text{Pl}(C) \leq 1$).

$$m_{|C}^{\text{Strong}}(B) = \begin{cases} m(B) / \text{Bel}(C) & \text{if } B \subseteq C \\ 0 & \text{otherwise} \end{cases}$$

$$m_{|C}^{\text{Weak}}(B) = \begin{cases} m(B) / \text{Pl}(C) & \text{if } B \cap C \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Notice that Dempster's rule also receives an interpretation in the credal context: it leads to a family consisting of the conditionals of those probabilities in the family which are the most likely (assessing a maximal probability to the event C we now know for sure) – hence the name “Max likelihood conditioning”.

So, in a credal game:

Definition 17. *The utility of a mixed strategy profile $\rho = (\rho_1, \dots, \rho_n)$, for player i of type θ_i , is:*

$$\text{XEU}_{(i, \theta_i)}(\rho) = \sum_{\sigma \in \Sigma} \left(\prod_{i \in N} \prod_{\theta_i \in \Theta_i} \rho_i(\theta_i)(\sigma_i(\theta_i)) \right) \times \sum_{B \in \mathcal{S}_{m_{|C}}} m_{|C}(B) \times f_{v_i^\sigma}^{\text{XEU}}(B)$$

where $\text{XEU} \in \{\text{JEU}, \text{CEU}\}$ and $m_{|C} \in \{m_{|C}^{\text{FH}}, m_{|C}^{\text{Strong}}, m_{|C}^{\text{Weak}}, m_{|C}^{\text{Dem}}\}$

This modelling leaves the definitions of pure and mixed Nash equilibria unchanged, i.e. ρ is a pure (resp. mixed) Nash equilibrium for XEU iff, whatever
380 (i, θ_i) , there exists no pure (resp. mixed) strategy ρ'_i with a greater XEU.

4. From Bel games to complete games

One of the most prominent results about Bayesian games is Howson's and Rosenthal's theorem [12]: any 2-player Bayesian game can be transformed into a (complete information) polymatrix game equivalent to the original one. This re-
385 sult is important from the computational point of view since it provides 2-player Bayesian games with practical resolution tools: to solve a 2-player Bayesian games, it is enough to use this theorem and to solve the resulting polymatrix game by using one of the algorithms proposed for such games [15, 16]. In the sequel, we generalize this theorem to Bel games and extend it to any number of
390 players.

4.1. The direct transform

A first idea is to define from a Bel game G , an hypergraphical game \tilde{G} , the vertices (players) of which are pairs (i, θ_i) with action set A_i – to each pure strategy σ of G corresponds a unique pure strategy $\tilde{\sigma}$ of \tilde{G} and conversely – we
395 call $\tilde{\sigma}$ the Selten³ transform of σ .⁴

Definition 18 (Selten transform of a pure strategy). *For any pure strategy σ of G , the Selten transform of σ is the vector $\tilde{\sigma}$ defined by $\tilde{\sigma}_{(i, \theta_i)} = \sigma(\theta_i)$.*

The local games of the hypergraphical game correspond to the focal elements of m . Roughly, (i, θ_i) plays in the local game corresponding to the focal element
400 B if the type θ_i is plausible for B – technically, if there exists $\theta' \in B$ such

³Named after Selten, who proposed this definition for Bayesian games [3].

⁴We could use the notation ρ for both, but the pure strategies of the Bel game are vectors of functions $\rho_i : \Theta_i \mapsto A_i$ while the pure strategies of \tilde{G} are vectors in $\prod_{i \in N} \prod_{\theta_i \in \Theta_i} A_i$. So, we keep the two notations $\tilde{\sigma}$ and σ .

that $\theta'_i = \theta_i$. In this local game, (i, θ_i) obtains a local utility $K_{|\theta_i} \cdot m(B) \cdot \min_{\theta' \in B, \theta'_i = \theta_i} u_i(\sigma(\theta'), \theta')$.

Given a profile of actions $\tilde{\sigma}$, and a player (i, θ_i) , the hypergraphical game sums (i, θ_i) 's local utilities over all the focal elements for which θ_i is plausible:
 405 the global utility for (i, θ_i) is equal to the XEU of the joint σ .

One may note that two pairs (i, θ_i) and (i, θ'_i) may play in the same local game – this happens when θ and θ' belong to the same focal element. In this case, the utility of (i, θ'_i) does not depend on the action played by (i, θ_i) and conversely.

410 For any focal element B of m , let $Players(B) = \{(i, \theta_i) \mid \theta \in B, i \in N\}$ – $Players(B)$ denotes the future players involved in the local game corresponding to B . Let \tilde{E} be the multiset $\tilde{E} := [Players(B) \mid B \in \mathcal{S}_m]$. The elements e of \tilde{E} and the focal elements in \mathcal{S}_m are in one-to-one correspondence and we denote B_e the focal element of m which leads to e . These notations allow us to
 415 propose a first, direct generalization of Howson's and Rosenthal's transform to Bel games:

Definition 19 (Direct transform of a Bel game). *The direct transform of a Bel game $G = (N, (A_i, \Theta_i, u_i)_{i \in N}, m)$ is the hypergraphical game $\tilde{G} = (\tilde{N}, \tilde{E}, (\tilde{A}_{(i, \theta_i)})_{(i, \theta_i) \in \tilde{N}}, (\tilde{u}_{(i, \theta_i)}^e)_{e \in \tilde{E}, (i, \theta_i) \in e})$ where:*

- 420 • $\tilde{N} = \{(i, \theta_i) \mid i \in N, \theta_i \in \Theta_i\}$,
- $\tilde{A}_{(i, \theta_i)} = A_i$,
- $\tilde{E} = [Players(B) \mid B \in \mathcal{S}_m]$,
- For each $e \in \tilde{E}$, $(i, \theta_i) \in e$ and $\tilde{\sigma} \in \tilde{A}$,
 $\tilde{u}_{(i, \theta_i)}^e(\tilde{\sigma}_e) = K_{|\theta_i} \cdot m(B_e) \cdot f_{\tilde{v}_i^{\tilde{\sigma}}}^{\text{XEU}}(B \cap \{\theta' \mid \theta'_i = \theta_i\})$, using $\tilde{v}_i^{\tilde{\sigma}}(\theta) = u_i(\tilde{\sigma}_\theta, \theta)$

425 It is easy to show that the XEU value of a pure strategy ρ in G and the global utility of $\tilde{\rho}$ in \tilde{G} are equal, whatever is the couple (i, θ_i) considered.

Proposition 1. *Let G be a Bel game based on the Dempster rule of conditioning and let \tilde{G} be its direct transform. For any pure strategy σ of G , it holds that $\text{XEU}_{(i, \theta_i)}(\sigma) = \tilde{u}_{(i, \theta)}(\tilde{\sigma})$.*

430 Let us extend the Selten transform to mixed strategies ρ of G : each $\tilde{\rho}_{(i, \theta_i)} = \rho_i(\theta_i)$ is then a probability distribution over A_i , and $\tilde{\rho}$ is then a vector of such distributions.

Proposition 2. *Let G be a Bel game and \tilde{G} its direct transform. For any mixed strategy ρ of G , it holds that $\text{XEU}_{(i, \theta_i)}(\rho) = \tilde{u}_{(i, \theta)}(\tilde{\rho})$.*

435 It can be checked that when m is a probability distribution, and G is a 2-player game, we get at most $|\Theta|$ local games, each involving two players (i, θ_i) and (j, θ_j) : \tilde{G} is a polymatrix game, and Howson's and Rosenthal's Theorem is recovered. More generally, we prove:

Theorem 1 (Generalized Howson-Rosenthal Theorem). *For any Bel game*
 440 *G based on a XEU utility and the Dempster rule of conditioning, there exists an hypergraphical game \tilde{G} such that ρ is a pure (resp. mixed) Nash equilibrium of G iff $\tilde{\rho}$ is a pure (resp. mixed) Nash equilibrium of \tilde{G} .*

Example 4. *Let us define the direct transform of the Bel game G corresponding to our running example (again, with Dempster conditioning and the Choquet*
 445 *expected utility). The set of players is: $\tilde{N} = \{(1, \bar{P}), (1, \bar{P}), (2, \bar{R}), (2, \bar{R})\}$. The set of actions are $\tilde{A}_{(i, \theta_i)} = \{P_i, Q_i, R_i\}$.*

Because m has two focal elements $B_1 = \{(\bar{P}, \bar{R})\}$ and $B_2 = \{(\bar{P}, \bar{R}), (P, \bar{R})\}$ each with a mass of $\frac{1}{2}$, \tilde{G} involves two local games. The set of players involved are respectively $e_1 = \{(1, \bar{P}), (2, \bar{R})\}$ and $e_2 = \{(1, \bar{P}), (1, P), (2, \bar{R})\}$. \tilde{G} 's hyper-
 450 *graph is drawn on Figure 1.*

Player $(2, \bar{R})$ plays only in e_2 , we have for instance:

$$\begin{aligned} \tilde{u}_{(2, \bar{R})}^{e_2}(R_1, P_1, R_2) &= K_{\bar{R}} \cdot m(B_2) \times \min \left[u_2((R_1, R_2), (P, \bar{R})), u_2((P_1, R_2), (\bar{P}, \bar{R})) \right] \\ &= 1 \times \min(2, 3) = 2. \end{aligned}$$

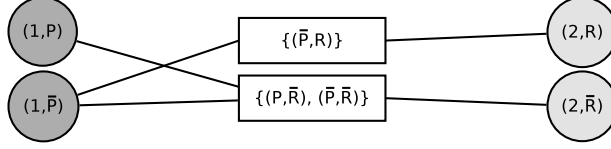


Figure 1: G 's direct transform. Gray circles denote vertices (players; one shade per agent), white boxes denote hyperedges (local games; linked to the players involved).

For player $(1, \bar{P})$, which plays in both local games, we have for instance:

$$\begin{aligned} \tilde{u}_{(1,\bar{P})}^{e_1}(P_1, P_2) &= K_{\bar{P}} \cdot m(B_1) \times u_1((P_1, P_2), (\bar{P}, R)) = 0.5 \times 2 = 1 \\ \tilde{u}_{(1,\bar{P})}^{e_2}(-, P_1, Q_2) &= K_{\bar{P}} \cdot m(B_2) \times u_1((P_1, Q_2), (\bar{P}, \bar{R})) = 0.5 \times 3 = 1.5 \end{aligned}$$

The Selten transform of the Nash equilibrium ρ' described in the example 3 is:

$$\tilde{\rho}'((1, \bar{P})) = P_1, \tilde{\rho}'((1, P)) = Q_1, \tilde{\rho}'((2, \bar{R})) = R_1, \tilde{\rho}'((2, R)) = Q_2.$$

It is easy to check that:

$$\tilde{u}_{(1,\bar{P})}(\tilde{\rho}') = \tilde{u}_{(1,\bar{P})}^{e_1}((P_1, Q_2)) + \tilde{u}_{(1,\bar{P})}^{e_2}((Q_1, P_1, R_2)) = \text{CEU}_{(1,\bar{P})}(\rho').$$

455 Notice that in the sum, one part of the utility of $(1, \bar{P})$ comes from subgame e_1 (i.e., from B_1) and the other part from local game e_2 (i.e., from B_2).

As to the complexity of the transform, let α (resp. β) be the maximum number of actions (resp. types) per player in G and k the degree of additivity of m . It holds that G contains n utility tables of size $(\alpha \cdot \beta)^n$ and the size of the description of m is bounded by $k \cdot n \cdot |\mathcal{S}_m|$. So, $\text{Size}(G)$ is in $O(n \cdot (\alpha \cdot \beta)^n + k \cdot n \cdot |\mathcal{S}_m|)$.

\tilde{G} contains $|\mathcal{S}_m|$ local games. Each of them involves at most $k \cdot n$ players (i, θ_i) – the size of their SNF representation is thus at most $k \cdot n \cdot \alpha^{kn}$ – hence a spatial cost for the representation of \tilde{G} in $O(|\mathcal{S}_m| \cdot k \cdot n \cdot \alpha^{kn})$. Notice now that 465 since m is k -additive, $|\mathcal{S}_m| < \beta^{k \cdot n}$. So, $\text{Size}(\tilde{G})$ is bounded by $k \cdot n \cdot (\alpha \cdot \beta)^{k \cdot n} \leq n^k \cdot (\alpha \cdot \beta)^{k \cdot n}$. In short, we get:

Proposition 3 (Complexity of the direct transform). *The direct transform of a Bel game G has a temporal complexity in $O(|\mathcal{S}_m| \cdot k^2 \cdot n \cdot \alpha^{k \cdot n} \cdot \beta) \subseteq O(k \cdot \beta \cdot \text{Size}(G)^k)$ and a spatial complexity in $O(|\mathcal{S}_m| \cdot k \cdot n \alpha^{k \cdot n}) \subseteq O(\text{Size}(G)^k)$*

470 So, the degree of additivity of the bpa is the main factor of complexity. Hopefully, low degrees of additivity can be assumed – it has indeed been shown [32, 33] that such low values (typically, $k \leq 3$) allow the description of many cases of interest. In such situations, the transform is quadratic or, at worst, cubic.

475 The direct transform, as defined above, holds for Demspster rule of conditioning. A variant can be used for each conditioning rule in which the focal elements of conditioned bpa are obtained by directly conditioning the original focal elements (each focal element B containing θ_i leads to a focal element $B_{|\theta_i}$). During strong or weak conditioning, masses aren't transferred but stay on the
 480 prior focal elements – i.e. $B_{|\theta_i} = B$. Modifying the local utility definition, switching the term $f_{\tilde{v}_i^{\sigma}}^{\text{XEU}}(B \cap \{\theta' \mid \theta'_i = \theta_i\})$ to $f_{\tilde{v}_i^{\sigma}}^{\text{XEU}}(B)$, captures both strong conditioning (with $K_{|\theta_i} = 1/\text{Pl}(\{\theta' \mid \theta'_i = \theta_i\})$) and weak conditioning (with $K_{|\theta_i} = 1/\text{Bel}(\{\theta' \mid \theta'_i = \theta_i\})$).

Unless in very particular cases, this kind of transform cannot be used with
 485 Fagin-Halpern's rule of conditioning, in which the conditioned focal elements cannot be assumed to be subsets of the prior ones.

The following transform enables both kind of conditioning.

4.2. The Conditioned transform

In the previously defined transform, we compute the CEU over the prior
 490 focal set, which is not possible in general. On the contrary, for the following transform, we first compute the set of conditioned focal elements, which will all lead to a local game, even if they are not (subsets of) prior focal elements.

Let $\mathcal{S}_{\cup} = \bigcup_{i \in N, \theta_i \in \Theta_i} \mathcal{S}_{m_{|\theta_i}}$ be the set of all $m_{|\theta_i}$'s focal elements, that is, the union of focal sets obtained after all possible conditioning “given θ_i ”. The local
 495 games of the hypergraphical game \tilde{G} correspond to the elements $B \in \mathcal{S}_{\cup}$. Again, (i, θ_i) plays in the local game corresponding to B if the type θ_i is plausible for B and obtains a local utility $m_{|\theta_i}(B) \times f^{\text{XEU}}(B)$, equal to the amount of XEU which is computed over B .

Definition 20 (Conditioned transform).

500

The conditioned transform of a Bel game $G = (N, (A_i, \Theta_i, u_i)_{i \in N}, m)$ is the hypergraphical game $\tilde{G} = (\tilde{N}, \tilde{E}, (\tilde{A}_{(i, \theta_i)})_{(i, \theta_i) \in \tilde{N}}, (\tilde{u}_{(i, \theta_i)}^e)_{e \in \tilde{E}, (i, \theta_i) \in e})$ where:

- $\tilde{N} = \{(i, \theta_i) \mid i \in N, \theta_i \in \Theta_i\},$
- $\tilde{A}_{(i, \theta_i)} = A_i,$
- 505 • $\tilde{E} = \left[\text{Players}(B) \mid B \in \bigcup_{(i, \theta_i) \in \tilde{N}} \mathcal{S}_{m_{|\theta_i}} \right],$
- For each $e \in \tilde{E}$, $(i, \theta_i) \in e$ and $\tilde{\rho} \in \tilde{A}$, $\tilde{u}_{(i, \theta_i)}^e(\tilde{\rho}_e) = m_{|\theta_i}(B_e) \times f_{\tilde{v}_i^{\tilde{\sigma}}}^{\text{XEU}}(B_e),$
where $\tilde{v}_i^{\tilde{\sigma}}(\theta) = u_i(\tilde{\sigma}_\theta, \theta).$

It is easy to show that the XEU value of a pure strategy ρ in G and the global utility of $\tilde{\rho}$ in \tilde{G} are equal, whatever is the couple (i, θ_i) considered. We also
510 prove that:

Proposition 4. Let G be a Bel game and \tilde{G} its conditioned transform. For any pure or mixed strategy ρ of G , it holds that:

- (i) $\text{CEU}_{(i, \theta_i)}(\rho) = \tilde{u}_{(i, \theta)}(\tilde{\rho})$
- (ii) ρ is a Nash equilibrium of G iff $\tilde{\rho}$ is a Nash equilibrium of \tilde{G} .

515 **Example 5.** The hypergraph of the conditioned transform our running example is drawn on Figure 2.

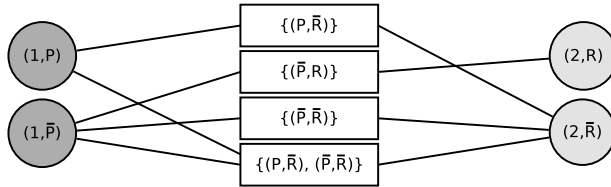


Figure 2: G 's conditioned transform. Gray circles are vertices (players; one color per agent), white boxes are hyperedges (local games; linked to the involved players).

This transform can be applied with any rule of conditioning. Notice that the hypergraphical game it leads to can be different from the one obtained with

the direct transform (assuming the same conditioning in both, e.g. Dempster's
 520 rule). In particular, it may produce more local games (with fewer players) than
 the direct transform.

Proposition 5 (Complexity of the conditioned transform).

*The conditioned transform of a Bel game G has a temporal complexity in $O(n \cdot \beta \cdot T_{cond} + |\mathcal{S}_\cup| \cdot k'^2 \cdot n \cdot \alpha^{k' \cdot n})$ and a spatial complexity in $O(|\mathcal{S}_\cup| \cdot k' \cdot n \cdot \alpha^{k' \cdot n})$, where
 525 $\mathcal{S}_\cup = \bigcup_{(i, \theta_i) \in \bar{N}} \mathcal{S}_{m|\theta_i}$, $k' = \max_{B \in \mathcal{S}_\cup} |B|$ and T_{cond} is the temporal complexity
 of a conditioning of m “given θ_i ”.*

*Using Dempster, Strong or Weak conditioning, it leads to a temporal complexity
 in $O(k \cdot \text{Size}(G)^k)$ and a spatial complexity of $O(\text{Size}(G)^k)$.*

So, the conditioned transform leads to a different hypergraphical game that
 530 the direct one, but has the same worst case spatial complexity. In practice, the
 size of the transformed game depends on the structure of the belief function.
 Typically, if a focal element B involves only one type θ_i for a given agent i ,
 both transforms will lead to the same local game $Players(B)$ (as $B = B_{|\theta_i}$),
 but the conditioned transform may produce (many) more local games and be
 535 less concise. If on the contrary, many types are compatible with a focal ele-
 ment B for any agent, the local game produced by the direct transform may
 have a bigger size. Consider for example a 2-player Bel game where m verify
 $m(\{(\theta_1, \theta_2), (\theta'_1, \theta'_2)\}) = m(\{(\theta_1, \theta'_2), (\theta'_1, \theta_2)\}) = 1/2$. With the direct trans-
 form, we get two 4-player local games, while the conditioned transform leads to
 540 four 2-player local games.

5. Conclusion

This article provides two main contributions. On the one hand, we define a
 model for simultaneous games of incomplete information based on belief func-
 tions. On the other hand, we introduce two transformations which make it
 545 possible to build an hypergraphical game (of complete information) equivalent
 to the initial Bel game, thus generalizing Howson and Rosenthal's theorem. The

transformation preserves utilities, so the study of a Bel game can be reduced to that of a complete game. In particular, Nash equilibria are in correspondence: any equilibrium in one game is an equilibrium in the other. Furthermore, under
550 some conditions (a low degree of additivity of the mass function), the transformation is polynomial in time and space; as a result, the algorithmic tools developed for hypergraphical games [16, 15] can be used to solve Bel games.

This work opens several research directions. First, we aim at extending this model to Choquet integrals based on any kind of capacity measure. It will allow
555 for the definition of games based on other decision rules, for instance the rank-dependent utility rule [34]. Then, a finer complexity analysis can be conducted, based on the characterization of the conditioned mass functions. Finally, we like to formalize those results with the COQ proof assistant [35] in order to build, with other in-progress results, a modular formal library on incomplete games
560 and decision theory.

Appendix A. Proofs

Appendix A.1. Proofs of correctness

Proof 1 (Relevancy of definitions 14, 16 and 17 – Utility of a mixed strategy profile).

On the one hand, any mixed strategy profile $\rho \in \prod_{i \in N} (\Theta_i \rightarrow \pi(A_i))$ defines a probability Pr^ρ over the possible pure strategy profiles $\sigma \in \Sigma = \prod_{i \in N} (\Theta_i \rightarrow A_i)$ by.⁵

$$\text{Pr}^\rho(\sigma) = \prod_{i \in N} \prod_{\theta_i \in \Theta_i} \rho_i(\theta_i)(\sigma_i(\theta_i))$$

*On the other hand, merging ρ and $m_{|\theta_i}$ leads to a bpa m^ρ over $A \times \Theta$, from which any element $X = \{(a, \theta), (a', \theta'), \dots\}$ is focal iff $B = \{\theta, \theta', \dots\}$ is focal
565 for m and (at least) one pure strategy profile σ is compatible with X and possible according to Pr^ρ ; i.e. $\sigma(\theta) = a$, $\sigma(\theta') = (a'), \dots$ and $\text{Pr}^\rho(\sigma) > 0$.*

⁵ $\pi(X)$ denotes the set of probabilities over X

Let $g.B := \{(g(\theta), \theta) \mid \theta \in B\}$ denote such focal element. By definition:

$$m_{|\theta_i}^\rho(g.B) = m(B) \times \sum_{\substack{\sigma \in \Sigma \\ \forall \theta \in B, \sigma(\theta) = g(\theta)}} \Pr^\rho(\sigma)$$

Thus, the XEU of a mixed strategy profile rewrites:

$$\begin{aligned} \text{XEU}_{(i, \theta_i)}(\rho) &= \sum_{g.B \in \mathcal{S}_{m_{|\theta_i}}^\rho} m_{|\theta_i}^\rho(g.B) \times f_{v_i^\sigma}^{\text{XEU}}(B) \\ &= \sum_{B \in \mathcal{S}_{m_{|\theta_i}}} \sum_{g: B \rightarrow A} m_{|\theta_i}(B) \times \left(\sum_{\substack{\sigma \in \Sigma \\ \forall \theta \in B, \sigma(\theta) = g(\theta)}} \Pr^\rho(\sigma) \right) \times f_{v_i^\sigma}^{\text{XEU}}(B) \\ &= \sum_{B \in \mathcal{S}_{m_{|\theta_i}}} \sum_{g: B \rightarrow A} \sum_{\substack{\sigma \in \Sigma \\ \forall \theta \in B, \sigma(\theta) = g(\theta)}} m_{|\theta_i}(B) \times \Pr^\rho(\sigma) \times f_{v_i^\sigma}^{\text{XEU}}(B) \end{aligned}$$

Given any $B \subseteq \Theta$, the set of functions $g: B \rightarrow A$ defines a partition of Σ , so:

$$\begin{aligned} \text{XEU}_{(i, \theta_i)}(\rho) &= \sum_{B \in \mathcal{S}_{m_{|\theta_i}}} \sum_{\sigma \in \Sigma} m_{|\theta_i}(B) \times \Pr^\rho(\sigma) \times f_{v_i^\sigma}^{\text{XEU}}(B) \\ &= \sum_{\sigma \in \Sigma} \Pr^\rho(\sigma) \times \sum_{B \in \mathcal{S}_{m_{|\theta_i}}} m_{|\theta_i}(B) \times f_{v_i^\sigma}^{\text{XEU}}(B) \\ &= \sum_{\sigma \in \Sigma} \left(\prod_{i \in N} \prod_{\theta_i \in \Theta_i} \rho_i(\theta_i)(\sigma_i(\theta_i)) \right) \times \sum_{B \in \mathcal{S}_{m_{|\theta_i}}} m_{|\theta_i}(B) \times f_{v_i^\sigma}^{\text{XEU}}(B) \end{aligned}$$

Proof 2 (Propositions 1 and 2 – Direct transform). Let G be a Bel Game, \tilde{G} be its conditioned transform and $E_{\theta_i} = \{\theta' \mid \theta'_i = \theta_i\}$ be the conditioning event “given θ_i ”.

570 Recall that for any mixed strategy profile ρ of G , $\tilde{\rho}$ is its Selten transform and $\tilde{\rho}_{(i, \theta_i)} = \rho_i(\theta_i)$ is a probability distribution over $A_i = \tilde{A}_{(i, \theta_i)}$. Similarly, for any pure strategy profile σ , $\tilde{\sigma}$ is its Selten transform and $\tilde{\sigma}_{(i, \theta_i)} = \sigma_i(\theta_i) \in A_i = \tilde{A}_{(i, \theta_i)}$. Finally, since Selten transform is bijective, we have:

$$\begin{aligned}
\text{XEU}_{(i,\theta_i)}(\rho) &= \sum_{\sigma \in \Sigma} \left(\prod_{i \in N} \prod_{\theta_i \in \Theta_i} \rho_i(\theta_i)(\sigma_i(\theta_i)) \right) \times \sum_{B \in \mathcal{S}_{m|_{\theta_i}}^{Dem}} m_{|\theta_i}^{Dem}(B) \times f_{v_i^\sigma}^{\text{XEU}}(B) \\
&= \sum_{\sigma \in \Sigma} \left(\prod_{i \in N} \prod_{\theta_i \in \Theta_i} \rho_i(\theta_i)(\sigma_i(\theta_i)) \right) \times \sum_{\substack{B \in \mathcal{S}_m \\ B \cap E_{\theta_i} \neq \emptyset}} K_{|\theta_i} \times m(B) \times f_{v_i^\sigma}^{\text{XEU}}(B \cap E_{\theta_i}) \\
&= \sum_{\sigma \in \Sigma} \left(\prod_{i \in N} \prod_{\theta_i \in \Theta_i} \rho_i(\theta_i)(\sigma_i(\theta_i)) \right) \times \sum_{\substack{e \in \tilde{E} \\ (i,\theta_i) \in e}} K_{|\theta_i} \times m(B_e) \times f_{v_i^\sigma}^{\text{XEU}}(B_e \cap E_{\theta_i}) \\
&= \sum_{\tilde{\sigma} \in \tilde{A}} \left(\prod_{(i,\theta_i) \in \tilde{N}} \tilde{\rho}_{(i,\theta_i)}(\tilde{\sigma}_{(i,\theta_i)}) \right) \times \tilde{u}_{(i,\theta_i)}(\tilde{\sigma}) \\
&= EU_{(i,\theta_i)}(\tilde{\rho})
\end{aligned}$$

Proof 3 (Proposition 4 – Conditioned transform). *Same remarks as for*

575 *proof 2. It leads to:*

$$\begin{aligned}
\text{XEU}_{(i,\theta_i)}(\rho) &= \sum_{\sigma \in \Sigma} \left(\prod_{i \in N} \prod_{\theta_i \in \Theta_i} \rho_i(\theta_i)(\sigma_i(\theta_i)) \right) \times \sum_{B \in \mathcal{S}_{m|_{\theta_i}}} m_{|\theta_i}(B) \times f_{v_i^\sigma}^{\text{XEU}}(B) \\
&= \sum_{\tilde{\sigma} \in \tilde{A}} \left(\prod_{(i,\theta_i) \in \tilde{N}} \tilde{\rho}_{(i,\theta_i)}(\tilde{\sigma}_{(i,\theta_i)}) \right) \times \sum_{B \in \mathcal{S}_{m|_{\theta_i}}} m_{|\theta_i}(B) \times f_{\tilde{v}_i^{\tilde{\sigma}}}^{\text{XEU}}(B) \\
&= \sum_{\tilde{\sigma} \in \tilde{A}} \left(\prod_{(i,\theta_i) \in \tilde{N}} \tilde{\rho}_{(i,\theta_i)}(\tilde{\sigma}_{(i,\theta_i)}) \right) \times \sum_{\substack{e \in \tilde{E} \\ \theta_i \in e}} m_{|\theta_i}(B_e) \times f_{\tilde{v}_i^{\tilde{\sigma}}}^{\text{XEU}}(B_e) \\
&= \sum_{\tilde{\sigma} \in \tilde{A}} \left(\prod_{(i,\theta_i) \in \tilde{N}} \tilde{\rho}_{(i,\theta_i)}(\tilde{\sigma}_{(i,\theta_i)}) \right) \times \tilde{u}_{(i,\theta_i)}(\tilde{\sigma}) \\
&= EU_{(i,\theta_i)}(\tilde{\rho})
\end{aligned}$$

Proof 4 (Theorem 1 – Extended Howson-Rosenthal’s theorem). *Direct corollary of proof 2 (also a corollary of proof 3).*

Appendix A.2. Proofs of complexity – mass function operations

In this section we consider a k -additive mass function m . We denote $s =$
580 $|\mathcal{S}_m|$. Set operations on focal elements (such as union, intersection and mem-

bership) involve $O(k)$ operations. The focal set and the corresponding masses involve a space of $O(ks)$.

We denote it by $Time(\cap) \in O(k)$ and $Size(m) \in O(ks)$ for example.

Proof 5 (Complexity of the Dempster, Strong and Weak conditioning “given C ”).

585 *First, note that every focal element of $m|_C^{Dem}$ is a subset of one of m , so $m|_C^{Dem}$ is k -additive and $|\mathcal{S}_{m|_C^{Dem}}| \leq s$. Thus we have $Size(m|_C^{Dem}) \leq Size(m) \in O(ks)$.*

To compute all the values of $m|_C^{Dem}$, two loops over \mathcal{S}_m suffice. Initialize $m|_C^{Dem}$ as a function which defaults to 0, and also a single variable $Pl(C) := 0$.

- *First, compute both $m|_C^{Dem}$'s unnormalized values and the normalization*
590 *factor $Pl(C)$: for each $B \in \mathcal{S}_m$, if $B \cap C \neq \emptyset$, add $m(B)$ to $m|_C^{Dem}(B \cap C)$ and to $Pl(C)$.*
- *Second, normalize those values: for each $B \in \mathcal{S}_{m|_C^{Dem}}$, $m|_C^{Dem}(B)$ becomes $m|_C^{Dem}(B) / Pl(C)$.*

The first loop involves s tests thus is in $O(ks)$. The second one doesn't involve
595 *any test and is thus in $O(s)$. Finally, $Time(m|_C^{Dem}) \in O(ks)$.*

For the weak conditioning, the proof is similar since the algorithm is almost identical: the only difference is that masses stays on B (they are not added to $B \cap C$). For the strong conditioning, the test condition changes to $B \subseteq C$ (thus the normalisation factor is $Bel(C)$), but stays in $O(k)$.

600 *Finally, $Time(m|_C^{Strong})$, $Time(m|_C^{Weak})$, $Size(m|_C^{Strong})$ and $Size(m|_C^{Weak})$ also are in $O(ks)$.*

Proof 6 (Complexity of the Fagin-Halpern conditioning “given C ”).

Following the algorithm of [20], it appears to be combinatorial to compute all values of $m|_C^{FH}$. Initialize $m|_C^{FH}$ by a function which defaults to 0 and also a
605 *single variable and $Pl(C) := 0$.*

- *First construct two subsets of \mathcal{S}_m , \mathcal{B}' and \mathcal{B}'' , which contains respectively the focal elements which are included in C and that intersect both C and C^c . It involves two tests in a single loop over \mathcal{S}_m , thus in $O(ks)$. One may also compute $Pl(C)$ during this loop, as for proof 5.*

- Then, for each $B_0 \in \mathcal{B}'$, consider all combinations B_1, \dots, B_m of elements of \mathcal{B}'' (there are $2^{|\mathcal{B}''|}$) and let $B^* = C \cap \bigcup_{i=0}^m B_i$. B^* 's computation involves $r \leq s$ set operations, thus in $O(ks)$. If $B^* \neq \emptyset$, then for all the $r!$ permutations p of $1..r$, add the value $\prod_{i=0}^m \frac{m(B_i)}{\text{Pl}(C) - \sum_{j=1}^i m(B_{p(j)})}$ to $m_C^{FH}(B^*)$, which involves $O(r(r-1)/2) = O(r^2)$ operations.

The second loop is the longest one: since $r \leq s$, it involves $O(s2^s(ks + ks!s^2))$. Finally, $\text{Time}(m_C^{FH}) \in O(ks^32^ss!)$.

Appendix A.3. Proofs of complexity – Games

In this section we consider a Bel game G with $n \geq 2$ players, each of those having at most $\alpha \geq 2$ actions and $\beta \geq 2$ types, among with a k -additive mass function with s focal elements, each of them is a set of n -tuples of types – so $s \leq \beta^{kn}$.

Proof 7 (Spatial complexity of a Bel game). *It holds that:*

- G contains n utility tables of size $(\alpha\beta)^n$ (one for each agent, assigning his/her utility to a strategy profile and a type configuration)
- The size of m is bounded by kns (each of the s focal elements contains at most k n -tuples of types)

Thus, $\text{Size}(G) \in O(n(\alpha\beta)^n + kns)$.

Proof 8 (Proposition 3 – Complexity of the direct transform). *The direct transform \tilde{G} of G has exactly s local games (one for each focal element B), in which players are possible pairs (i, θ_i) such as $\exists \theta' \in B, \theta'_i = \theta_i$. They may be kn such pairs, so the corresponding local game is described by at most kn matrices of α^{kn} cells, hence a spatial cost for the representation of \tilde{G} in $O(skn\alpha^{kn})$. Recall that $s \leq \beta^{kn}$ and $kn \leq n^k$, it holds that $\text{Size}(\tilde{G})$ is bounded by $kn(\alpha\beta)^{kn} \leq n^k(\alpha\beta)^{kn}$, i.e., $\text{Size}(\tilde{G}) \in O(\text{Size}(G)^k)$*

To instantiate those matrices, one has to compute each of the utility values as from the definition 19:

- First, for each of the $n\beta$ pairs (i, θ_i) , compute $\text{Pl}(E_{\theta_i})$ by a single loop over m 's focal set in which $n\beta$ tests are made, so it involves $O(skn\beta)$ operations.
- 640 • Then, for each of the s focal elements B (i.e. a local game e), for each of the kn possible corresponding pairs (i, θ_i) and for each of the α^{kn} possible local strategy profiles $\tilde{\sigma}_e$, set $\tilde{u}_{(i, \theta_i)}^e(\tilde{\sigma}_e) := m(B) \times f_{\tilde{\sigma}_e}^{\text{XEU}}(B \cap E_{\theta_i}) / \text{Pl}(E_{\theta_i})$, where $f^{\text{XEU}}(b)$ involves k operations.

Thus we have $\text{Time}(\tilde{G}) \in O(skn\beta + sk^2n\alpha^{kn}) = O(skn(\beta + k\alpha^{kn}))$. Since m is k -additive, $\text{Time}(\tilde{G})$ is bounded by $kn\beta^{kn}(\beta + k\alpha^{kn}) \in O(k^2n\beta(\alpha\beta)^{kn}) \subseteq O(\beta kn^k(\alpha\beta)^{kn})$; i.e. $\text{Time}(\tilde{G}) \in O(\beta k \times \text{Size}(G)^k)$.

Note that since the normalization doesn't change the equilibria of \tilde{G} , usually the first loop is not necessary and the complexity becomes $\text{Time}(\tilde{G}) \in O(k \times \text{Size}(G)^k)$.

650 **Proof 9 (Proposition 5 – Complexity of the conditioned transform).**

Let $s' = |\mathcal{S}_\cup|$ be the total number of focal elements, after all conditioning, and $k' = \max_{B \in \mathcal{S}_\cup} |B|$ their maximal size. The conditioned transform \tilde{G} of F has exactly s' local games, which involve at most $k'n$ players each, thus they are described by at most $k'n$ matrices of $\alpha^{k'n}$ cells, hence a spatial cost for the representation of \tilde{G} in $O(k'ns'\alpha^{k'n})$.

If the conditioning is one of the Dempster, Strong or Weak ones, it holds that $k' \leq k$ and $s' \leq \beta^{kn}$, so the bound becomes $kn(\alpha\beta)^{kn}$; i.e. $\text{Size}(\tilde{G}) \in O(\text{Size}(G)^k)$.

On the contrary, if the Fagin-Halpern conditioning is used, we can just bound $k' \in O(n\beta)$ and $s' \in O(2^{n\beta})$, and thus get a spatial complexity $\text{Size}(\tilde{G}) \in O(n^2\beta 2^{n\beta} \alpha^{n^2\beta})$.

To construct those local utility matrices, one has to compute each of the utility values:

- First, for each of the $n\beta$ pairs (i, θ_i) , compute $m_{|\theta_i}$, according to the chosen conditioning, say it costs T_{cond} .

- Then, for each of the s' local games, for each of its $k'n$ players and for each of the $\alpha^{k'n}$ possible local strategy profiles $\tilde{\sigma}_e$, set $\tilde{u}_{(i,\theta_i)}^e(\tilde{\sigma}_e) := m_{|\theta_i}(B_e) \times f_{\tilde{v}_i^{\tilde{\sigma}}}^{XEU}(B_e)$, where $f_{\tilde{v}_i^{\tilde{\sigma}}}^{XEU}(B_e)$ involves at most k' operations.

The first loop costs $n\beta T_{cond}$ operations, the second one costs $s'k'^2n\alpha^{k'n}$ operations.
670 tions.

If the conditioning is one of the Dempster, Strong or Weak ones, it holds that $k' \leq k$, $s' \leq \beta^{kn}$ and $T_{cond} \in O(kn)$, so the bound becomes $k^2n(\alpha\beta)^{kn}$; i.e. $Time(\tilde{G}) \in O(k \cdot Size(G)^k)$.

On the contrary, if the Fagin-Halpern conditioning is used, we can just bound
675 $k' \in O(n\beta)$, $s' \in O(2^{n\beta})$ and $T_{cond} \in O(ks^32^s s!)$, and thus get a temporal complexity $Time(\tilde{G}) \in O(n\beta ks^32^s s! + n^3\beta^22^{n\beta}\alpha^{n^2\beta})$.

References

- [1] O. Morgenstern, J. Von Neumann, Theory of Games and Economic Behavior, Princeton University Press, 1953.
- 680 [2] J. Nash, Non-Cooperative Games, Annals of Mathematics (1951) 286–295.
- [3] J. C. Harsanyi, Games With Incomplete Information Played by “Bayesian” Players, I–III Part I. The Basic Model, Management Science 14 (3) (1967) 159–182.
- [4] D. Ellsberg, Risk, Ambiguity, and the Savage Axioms, The Quarterly Journal of Economics (1961) 643–669.
685
- [5] A. P. Dempster, Upper and Lower Probabilities Induced by a Multivalued Mapping, The Annals of Mathematical Statistics 38 (1967) 325–339.
- [6] G. Shafer, A Mathematical Theory of Evidence, Princeton University Press, 1976.
- 690 [7] G. Choquet, Theory of Capacities, in: Annales de l’institut Fourier, Vol. 5, 1954, pp. 131–295.

- [8] I. Gilboa, D. Schmeidler, Maxmin Expected Utility with Non-Unique Prior, *Journal of Mathematical Economics* 18 (2) (1989) 141–153.
- [9] D. Schmeidler, Integral Representation without Additivity, *Proceedings of the American Mathematical Society* 97 (2) (1986) 255–261.
- [10] P. Smets, R. Kennes, The Transferable Belief Model, *Artificial Intelligence* 66 (2) (1994) 191–234.
- [11] J.-Y. Jaffray, Linear Utility Theory for Belief Functions, *Operations Research Letters* 8 (2) (1989) 107–112.
- [12] J. T. Howson Jr, R. W. Rosenthal, Bayesian Equilibria of Finite Two-Person Games with Incomplete Information, *Management Science* 21 (3) (1974) 313–315.
- [13] E. Yanovskaya, Equilibrium Points in Polymatrix Games, *Lithuanian Mathematical Journal* 8 (1968) 381–384.
- [14] C. H. Papadimitriou, T. Roughgarden, Computing Correlated Equilibria in Multi-Player Games, *Journal of the Association for Computing Machinery* 55 (3) (2008) 1–29.
- [15] M. Wahbi, K. N. Brown, A Distributed Asynchronous Solver for Nash Equilibria in Hypergraphical Games, in: *Frontiers in Artificial Intelligence and Applications*, Vol. 285, 2016, pp. 1291–1299.
- [16] A. Chapman, A. Farinelli, E. M. de Cote, A. Rogers, N. Jennings, A Distributed Algorithm for Optimising over Pure Strategy Nash Equilibria, in: *Proceedings of the AAAI Conference on Artificial Intelligence*, Vol. 24, 2010, pp. 749–755.
- [17] D. Dubois, H. Prade, On the Unicity of Dempster Rule of Combination, *International Journal of Intelligent Systems* 1 (2) (1986) 133–142.
- [18] P. Walley, *Statistical Reasoning with Imprecise Probabilities*, Chapman & Hall, 1991.

- [19] J. Y. Halpern, R. Fagin, Two Views of Belief: Belief as Generalized Probability and Belief as Evidence, *Artificial intelligence* 54 (3) (1992) 275–317.
- [20] J.-Y. Jaffray, Bayesian Updating and Belief Functions, *IEEE Transactions on Systems, Man, and Cybernetics* 22 (5) (1992) 1144–1152.
- [21] D. Dubois, T. Denoeux, Conditioning in Dempster-Shafer Theory: Prediction vs. Revision, in: *Belief Functions: Theory and Applications - Proceedings of the 2nd International Conference on Belief Functions*, Springer, 2012, pp. 385–392.
- [22] R. Fagin, J. Y. Halpern, A New Approach to Updating Beliefs, in: *Proceedings of the Sixth Annual Conference on Uncertainty in Artificial Intelligence*, 1990, pp. 347–374.
- [23] L. J. Savage, *The Foundations of Statistics*, Wiley, 1954.
- [24] I. Gilboa, Expected Utility with Purely Subjective Non-Additive Probabilities, *Journal of Mathematical Economics* 16 (1) (1987) 65–88.
- [25] D. Schmeidler, Subjective Probability and Expected Utility without Additivity, *Econometrica: Journal of the Econometric Society* (1989) 571–587.
- [26] I. Gilboa, D. Schmeidler, Additive Representations of Non-Additive Measures and the Choquet Integral, *Annals of Operations Research* 52 (1) (1994) 43–65.
- [27] M. J. Kearns, M. L. Littman, S. P. Singh, Graphical Models for Game Theory, in: J. S. Breese, D. Koller (Eds.), *Proceedings of the 17th Conference in Uncertainty in Artificial Intelligence*, 2001, pp. 253–260.
- [28] R. B. Myerson, *Game Theory*, Harvard university press, 2013.
- [29] J.-Y. Jaffray, Linear Utility Theory and Belief Functions: a Discussion, in: *Progress in decision, utility and risk theory*, Springer, 1991, pp. 221–229.

- [30] P. Smets, Jeffrey’s Rule of Conditioning Generalized to Belief Functions,
 745 in: Uncertainty in artificial intelligence, Elsevier, 1993, pp. 500–505.
- [31] B. Planchet, Credibility and Conditioning, Journal of Theoretical Probability 2 (3) (1989) 289–299.
- [32] P. Miranda, M. Grabisch, P. Gil, Dominance of Capacities by k -Additive Belief Functions, European Journal of Operational Research 175 (2) (2006)
 750 912–930.
- [33] M. Grabisch, Upper Approximation of Non-additive Measures by k -additive Measures—The Case of Belief Functions., in: Proceedings of the First International Symposium on Imprecise Probabilities and Their Applications, 1999, pp. 158–164.
- 755 [34] J. Quiggin, A Theory of Anticipated Utility, Journal of economic behavior & organization 3 (4) (1982) 323–343.
- [35] The Coq Development Team, The Coq Proof Assistant: Reference Manual: version 8.15 (2022).
 URL <https://coq.github.io/doc/V8.15.0/refman/>