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Abstract. We investigate interior and exterior overdetermined boundary value problems on triangles, which corresponds to study stationary triangles for variational functionals under volume or perimeter constraint. We prove that in certain cases the only triangle supporting solutions is the equilateral one. In some other cases, we obtain that all triangles support solutions, thus extending (through a simpler proof) what recently shown in [8].

Keywords: overdetermined problems, triangles, torsional rigidity, principal frequency, p-capacity.

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1. Introduction

Starting from the celebrated symmetry result by Serrin [26], a huge literature is devoted to the study of overdetermined boundary value problems of various kind, both for interior and exterior problems, and for a large variety of differential operators and overdetermined conditions. With no attempt of completeness, see [1, 4, 5, 6, 10, 12, 13, 14, 16, 20, 24, 25, 27, 28, 29] and references therein.

In this setting we feel a bit surprising that no symmetry result is available, to the best of our knowledge, for overdetermined problems set on polygons. This seems to be an interesting topic especially from the point of view of Calculus of Variations. In fact, polygons which support solutions to certain overdetermined problems turn out to be stationary, in the sense of shape derivation, for some some long-standing open problems in shape optimization; among these, let us mention the minimization of the principal frequency among polygons with a given number of sides under an area constraint, for which the regular gon is conjectured to be optimal [18, Section 3.3], or the minimization of the p-capacity among convex planar sets under a perimeter constraint, for which the expected solution is a degenerate polygon, *i.e.* a line segment [7, Remark 2.4].

In this paper we give a first contribution to the study of overdetermined problems on polygons, by considering the case of triangles. As a model case, we start from the torsion problem as in Serrin's paper, namely we consider the Dirichlet problem

$$-\Delta u = 1 \quad \text{in} \quad T, \qquad u = 0 \quad \text{on} \quad \partial T, \tag{1.1}$$

where we fix the space dimension to be $n = 2$, and T lies in the class $\mathcal{T}$ of (nondegenerate) triangles. We overdetermine problem (1.1) with one of the following setting of boundary conditions of integral type for $|\nabla u|$ on ∂T:

$$\int_{A_i}^{A_{i+1}} \left(\frac{\ell_i}{2} - |x - A_i| \right) |\nabla u(x)|^2 \, dx = 0 \qquad \forall i \in \{1,2,3\}\,, \tag{1.2}$$

$$\int_{A_i}^{A_{i+1}} \left(\frac{\ell_i}{2} - |x - A_i| \right) |\nabla u(x)|^2 \, dx = \kappa \frac{\ell_i}{2} \big(f(\theta_i) - f(\theta_{i+1}) \big) \qquad \forall i \in \{1,2,3\}\,, \tag{1.3}$$

where A_i are the vertices of T, ℓ_i is the length of the side $[A_i A_{i+1}]$, κ is a positive constant, and

$$f(\theta) := \cot\theta + \frac{1}{\sin\theta}\,. \tag{1.4}$$

Conditions (1.2) and (1.3) are very natural candidates to yield symmetry. Indeed, as we will show in Section 2, they correspond precisely to the stationarity conditions for the torsional rigidity respectively under a volume and a perimeter constraint, when one rotates a side of T around its midpoint. Since the equilateral triangle is known to be the unique maximizer of torsional rigidity among triangles of given volume or perimeter, it is reasonable to expect it may be also the unique "critical" triangle.

These arguments can be repeated if the torsion functional is replaced by the principal frequency, namely we may consider in place of (1.1) the Dirichlet problem

$$-\Delta u = \lambda_1(T)u \quad \text{in} \quad T, \qquad u = 0 \quad \text{on} \quad \partial T, \qquad \int_T u^2\,dx = 1, \tag{1.5}$$

where $\lambda_1(T)$ is the first Dirichlet eigenvalue of the laplacian.

For both problems (1.1) and (1.5), we prove the following Serrin-type result:

Theorem 1.1 (triangular symmetry, interior case)**.** *Let $T \in \mathcal{T}$, and let u be the unique solution to* (1.1) *or to* (1.5)*. Then:*

- *u satisfies the overdetermined boundary condition* (1.2) *if and only if T is equilateral.*
- *u satisfies the overdetermined boundary condition* (1.3) *for some $\kappa \geq 0$ if and only if T is equilateral.*

Our approach is based on a reflection argument, and is reminiscent of Serrin's proof in the use of Hopf's boundary point principle. In particular, similarly as shown by Reichel in [25] for Serrin's result, our symmetry statement can be extended to the case of exterior problems. However, this extension is a partial one, in the sense that it only concerns the overdetermined boundary condition (1.2). To be more precise, for a fixed exponent $p \in (1,2)$, denote by Δ_p the p-laplace operator, and consider the p-capacitary problem

$$\Delta_p u = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus T, \qquad u = 1 \quad \text{on} \quad \partial T, \qquad \lim_{|x|\to\infty} u(x) = 0. \tag{1.6}$$

We have:

Theorem 1.2 (triangular symmetry, exterior case)**.** *Let $T \in \mathcal{T}$, and let u be the unique solution to* (1.6)*. Then u satisfies the overdetermined boundary condition* (1.2) *(with $|\nabla u|^2$ replaced by $|\nabla u|^p$) if and only if T is equilateral.*

Remark 1.3. During the proof of Theorems 1.1 and 1.2, the following stronger fact will be shown: if u is the unique solution to (1.1), or (1.5), or (1.6), as soon as $\theta_i < \theta_{i+1}$ the integral at the l.h.s. of (1.2) or (1.3) turns out to have a sign, precisely it is strictly negative in case of the interior problems, and strictly positive for the capacitary one.

Remark 1.4 (partially overdetermined problems). As a consequence of Remark 1.3, we get that if the overdetermined condition (1.2) or (1.3) holds only on one side of the triangle, then the angles adjacent to such a side must be equal, so that the triangle is isosceles with the selected side as a basis. Thus, in order to obtain that T is equilateral, it is enough to impose the overdetermined condition (1.2) (respectively, (1.3)) on two sides of the triangle. This refinement can be interpreted as a symmetry result for a partially overdetermined boundary value problem, in the spirit of [13].

Remark 1.5. Problem (1.6) overdetermined by (1.3) (with $|\nabla u|^2$ replaced by $|\nabla u|^p$) remains interestingly open. This discrepancy with respect to the overdetermined problems (1.1)-(1.3) and (1.5)-(1.3) covered by Theorems 1.1 should not surprise the attentive reader. Indeed, while condition (1.2) represents a "discrete" analogous to the stationarity condition under volume constraint which for smooth domains reads $|\nabla u| = c$, condition (1.3) should be assimilated to the stationarity condition under perimeter constraint which for smooth domains reads $|\nabla u| = c(H)$, being c a function of the boundary mean curvature H. This latter overdetermined condition, treated by Serrin in [26, Theorem 3], can be successfully handled by the moving planes method only under a specific monotonicity assumption on c, and the favourable sign of the monotonicity changes when passing from interior to exterior problems.

As a second case of study, we overdetermine problems (1.1), or (1.5), or (1.6), with a different setting of boundary conditions, which looks as well quite natural, as they correspond to the stationarity conditions for the associated energy functional, respectively under a volume and a perimeter constraint, when one makes a side of T move in a parallel way to itself (*cf.* Section 2). They read

$$\int_{A_i}^{A_{i+1}} |\nabla u(x)|^2 \, dx = \kappa \ell_i \qquad \forall i \in \{1,2,3\}\,, \tag{1.7}$$

$$\int_{A_i}^{A_{i+1}} |\nabla u(x)|^2 \, dx = \kappa\big(f(\theta_i) + f(\theta_{i+1})\big) \qquad \forall i \in \{1,2,3\}\,, \tag{1.8}$$

where κ is a positive constant, $f(\theta)$ is given by (1.4), and as above we intend that the term $|\nabla u(x)|^2$ is changed into $|\nabla u(x)|^p$ when dealing with problem (1.6).

In contrast to conditions (1.2)-(1.3), conditions (1.7)-(1.8) do not yield symmetry. The difference is in fact even more drastic, since the last setting of conditions turns out to be always satisfied. We have indeed:

Theorem 1.6 (triangular equidistribution)**.** *Let $T \in \mathcal{T}$, let u be the unique solution to* (1.1), (1.5)*, or* (1.6)*, and let F be respectively the torsional rigidity, the first Dirichlet eigenvalue of the Laplacian, and the p-capacity. Then*

- *u satisfies* (1.7)*, with* $\kappa = \dfrac{|\alpha|}{2}\dfrac{F(T)}{\mathrm{Vol}(T)}$
- *u satisfies* (1.8)*, with* $\kappa = |\alpha|\dfrac{F(T)}{\mathrm{Per}(T)}$,

where α is the homogeneity degree of F under domain dilation (divided by $(p-1)$ in case of problem (1.6)*).*

Only in the case of the first Laplacian Dirichlet eigenvalue and just for condition (1.7), Theorem 1.6 was proved by Christianson in the recent paper [8], by using a completely different approach. Our proof seems to be more direct, and has the advantage to work the same way in all the cases covered by the statement, and possibly even in more general situations, such as for instance the cases of p-torsion function and first Dirichlet eigenvalue of the p-Laplacian. (Avoiding the technicalities related to more sophisticated versions of maximum principles is the reason why we preferred to present problems (1.1) and (1.5) in their linear version).

To extend our results to quadrilaterals or polygons with an arbitrary number of sides seems to require some different idea and remains by now a challenging open problem.

2. Stationarity conditions

In this section we consider scale invariant energies defined on the class $\mathcal{K}^2$ of convex bodies with nonempty interior in $\mathbb{R}^2$, and we show that the equalities (1.2)-(1.3) and (1.7)-(1.8) correspond respectively to the stationarity conditions for such energies, when the perturbed domain is a triangle and the perturbation consists in rotating one side around its midpoint and in moving it in a parallel way to itself.

To be more precise, we fix the following setting:

Definition 2.1 (energy functionals)**.** *Let $\Omega \in \mathcal{K}^2$. Let F be either the torsional rigidity, or the first eigenvalue of the Laplacian, or the p-capacity ($1 < p < 2$), defined respectively as*

$$\tau(\Omega) := -\inf_{u \in H^1_0(\Omega)} \int_\Omega \big(|\nabla u|^2 - 2\,u\big)\,dx\,, \qquad \lambda_1(\Omega) := \inf_{u \in H^1_0(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2\,dx}{\int_\Omega |u|^2\,dx}\,,$$

$$\mathrm{Cap}_p(\Omega) := \inf\Big\{ \int_{\mathbb{R}^2} |\nabla u|^p\,dx \;:\; u \in \dot{W}^{1,p}(\mathbb{R}^2)\,,\;\; u = 1 \text{ on } \Omega \Big\}\,,$$

where $\dot{W}^{1,p}(\mathbb{R}^2)$ denotes the closure of space of the space of smooth compactly supported functions with respect to the norm $\int_{\mathbb{R}^2}|\nabla u|^p\,dx$.
We denote by α the homogeneity degree of F under dilations, so that

$$\alpha=\begin{cases}4 & \text{if } F=\tau\\ -2 & \text{if } F=\lambda_1\\ 2-p & \text{if } F=\mathrm{Cap}_p\ .\end{cases}$$

Moreover, we denote by u_Ω respectively either the torsion function, or the first Dirichlet eigenfunction of the Laplacian, of the p-capacitary potential of Ω, namely the solution to the Euler-Lagrange equation of F given in (1.1), (1.5), (1.6).

Next we introduce the two family of deformations we are going to consider for triangles.

Definition 2.2 (rotation around the midpoint)**.** *Given $T\in\mathcal{T}$ with vertices A_1, A_2 and A_3, and $t\in\mathbb{R}$ small enough, we denote by $\{T_t\}$ the triangle with vertices A_1^t, A_2^t, A_3^t, obtained by keeping fixed $A_3=:A_3^t$ and replacing A_1 and A_2 by A_1^t and A_2^t in such a way that (see Figure 1):*

- *the line determined by A_1^t and A_2^t is a rotation on an angle t of the line determined by A_1 and A_2 around the midpoint M of the side $[A_1,A_2]$;*
- *A_1^t and A_2^t lie respectively on the lines determined by A_3, A_1 and A_3, A_2;*
- *the direction of the rotation is determined by the convention that, for $t>0$, A_2^t lies on the segment $[A_2,A_3]$, while for $t<0$, A_1^t lies on the segment $[A_1,A_3]$.*

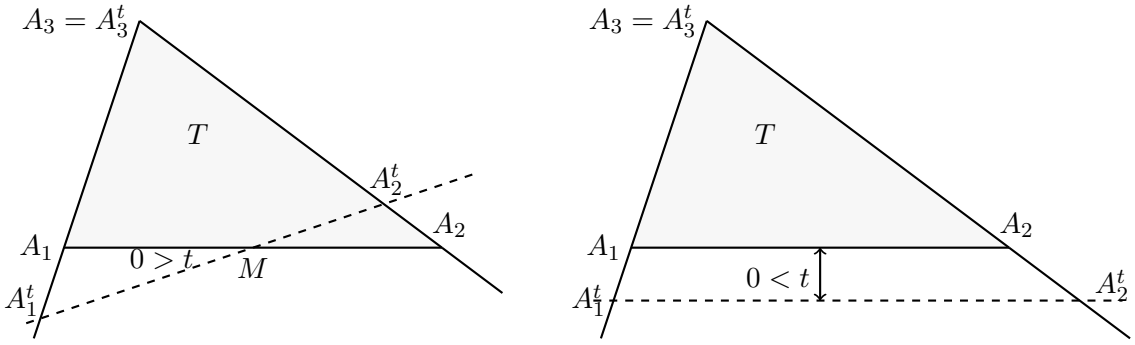

FIGURE 1. The rotating around the midpoint (on the left) and the parallel movement of a side (on the right).

Definition 2.3 (parallel movement)**.** *Given $T\in\mathcal{T}$ with vertices A_1, A_2 and A_3, and $t\in\mathbb{R}$ small enough, we denote by $\{T_t\}$ the triangle with vertices A_1^t, A_2^t, A_3^t, obtained by keeping fixed $A_3=:A_3^t$ and replacing A_1 and A_2 by A_1^t and A_2^t in such a way that (see Figure ...):*

- *the line determined by A_1^t and A_2^t is parallel to the side $[A_1,A_2]$ and at distance $|t|$ from $[A_1,A_2]$;*
- *A_1^t and A_2^t lie respectively on the lines determined by A_3, A_1 and A_3, A_2;*
- *the direction of the movement is determined by the convention that, for $t>0$, A_2^t and A_1^t do not lie to the segment $[A_2,A_3]$ and to the segment $[A_1,A_3]$ respectively.*

We are now ready to identify stationary triangles:

Proposition 2.4. *Let $\mathcal{E}_{\mathrm{Vol}}(T)$ and $\mathcal{E}_{\mathrm{Per}}(T)$ be defined by*

$$\mathcal{E}_{\mathrm{Vol}}(\Omega):=\frac{F(\Omega)^{\frac{1}{\alpha}}}{\mathrm{Vol}(\Omega)^{1/2}}\,,\qquad \mathcal{E}_{\mathrm{Per}}(\Omega):=\frac{F(\Omega)^{\frac{1}{\alpha}}}{\mathrm{Per}(\Omega)}\,,\tag{2.1}$$

with F as in Definition 2.1.

(i) *A triangle $T \in \mathcal{T}$ is stationary for $\mathcal{E}_{\mathrm{Vol}}(T)$ and $\mathcal{E}_{\mathrm{Per}}(T)$ with respect to the perturbations $\{T_t\}$ as in Definition* 2.2*, if and only if u_T satisfies respectively* (1.2) *and* (1.3).

(ii) *A triangle $T \in \mathcal{T}$ is stationary for $\mathcal{E}_{\mathrm{Vol}}(T)$ and $\mathcal{E}_{\mathrm{Per}}(T)$ with respect to the perturbations $\{T_t\}$ as in Definition* 2.3*, if and only if u_T satisfies respectively* (1.7) *and* (1.8).

The above statement is obtained as a direct consequence of Lemmas 2.6 and Lemma 2.7 below. In turn, to obtain such results, we need a more general first derivation formula, which is valid for suitable perturbations of convex domains, as stated in Lemma 2.5 below. Let us recall that, for any $\Omega \in \mathcal{K}^2$, the gradient of the function u_Ω in Definition 2.1 is well defined on $\partial\Omega$ and belongs to $L^p(\partial\Omega)$ [11, 21]. *Here and throughout the remaining of the paper, we mean that the exponent p equals* 2 *when dealing with the interior problems* (1.1) *and* (1.5).

Lemma 2.5 (Shape derivatives with respect to generic perturbations)**.** *Let $\Omega \in \mathcal{K}^2$, with unit outer normal n_Ω, and let $\Omega_t = \Phi_t(\Omega)$, where $t \in [0,T) \to \Phi_t \in W^{1,\infty}(\mathbb{R}^2)$ is differentiable at $t = 0$, with $\Phi_0(x) = x$ and $\frac{d}{dt}\big|_{t=0}\Phi_t(x) = V(x)$. Let F and u_Ω be as in Definition* 2.1*. Then*

$$\frac{d}{dt}\Big|_{t=0} F(\Omega_t) = (\operatorname{sign}\alpha)\,(p-1)\int_{\partial\Omega} V\cdot n_\Omega |\nabla u_\Omega|^p \, d\mathcal{H}^1\,. \tag{2.2}$$

Proof. When $F = \tau$, the statement can be deduced by applying the duality approach developed in [2] for shape derivative of minima of integral functionals (*cf.* in particular [2, Theorems 3.7 and Example 3.9], and see also [3, Section 7.2]).

When $F = \lambda_1$, we refer to [19, Section 5.9.3].

When $F = \mathrm{Cap}_p$, one can adapt the duality approach in [2], by showing that [2, Theorems 3.3 and 3.7] continue to hold for the exterior problem of p-capacity and give directly (2.2). However, for the sake of completeness, we give here a formal derivation of (2.2), by sketching a proof along the more classical line adopted in [15] to compute the shape derivative ot the 2-capacity of convex bodies in $\mathbb{R}^3$.

Let u_t be the p-capacitary potential on Ω_t, that is

$$-\mathrm{div}(|\nabla u_t|^{p-2}\nabla u_t) = 0 \quad \text{in} \quad \mathbb{R}^2\setminus\Omega_t, \qquad u_t = 1 \quad \text{on} \quad \partial\Omega_t.$$

By arguing in a similar way as in the Appendix of [15], we see that the map $\mathbb{R} \ni t \mapsto u_t \in W^{1,p}_{loc}(\mathbb{R}^2\setminus\overline{\Omega})$ is differentiable in $t = 0$. Denoting by u' the derivative of u_t at $t = 0$ and formally differentiating the equation for u_t, we get that u' is a solution of

$$\begin{cases} -\mathrm{div}\big(|\nabla u|^{p-2}\nabla u' + (p-2)|\nabla u|^{p-4}(\nabla u\cdot\nabla u')\nabla u\big) = 0 \quad \text{in} \quad \mathbb{R}^2\setminus\Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad u' = -V\cdot\nabla u \quad \text{on} \quad \partial\Omega. \end{cases} \tag{2.3}$$

Let now $\psi \in C^\infty_c(\mathbb{R}^2)$ be a smooth function such that $\psi \equiv 1$ in a neighborhood of $\overline{\Omega}$. Testing (2.3) with $u - \psi$ and integrating by parts we get

$$\begin{aligned} \int_{\Omega^c} |\nabla u|^{p-2}\nabla u'\cdot\nabla(u-\psi)\,dx &= -(p-2)\int_{\Omega^c} |\nabla u|^{p-4}(\nabla u\cdot\nabla u')(\nabla u\cdot\nabla(u-\psi))\,dx \\ &= -(p-2)\int_{\Omega^c} |\nabla u|^{p-2}(\nabla u\cdot\nabla u')\,dx \\ &\quad + (p-2)\int_{\Omega^c} |\nabla u|^{p-4}(\nabla u\cdot\nabla u')(\nabla u\cdot\nabla\psi)\,dx, \end{aligned}$$

which gives

$$(p-1)\int_{\Omega^c} |\nabla u|^{p-2}\nabla u'\cdot\nabla u\,dx = \int_{\Omega^c}\Big(|\nabla u|^{p-2}\nabla u'\cdot\nabla\psi + (p-2)|\nabla u|^{p-4}(\nabla u\cdot\nabla u')(\nabla u\cdot\nabla\psi)\Big)\,dx. \tag{2.4}$$

On the other hand, denoting by n_t the unit outer normal to Ω_t, we have

$$\mathrm{Cap}_p(\Omega_t) = \int_{\Omega_t^c} |\nabla u_t|^p\,dx = -\int_{\partial\Omega_t} |\nabla u_t|^{p-2}\frac{\partial u_t}{\partial n_t}\psi\,d\mathcal{H}^1 = \int_{\Omega_t^c} |\nabla u_t|^{p-2}\nabla u_t\cdot\nabla\psi\,dx.$$

Taking the derivative in $t=0$, exlpoiting (2.4) and integrating by parts, we obtain

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\mathrm{Cap}_p(\Omega_t) &= \int_{\Omega^c}\Big(|\nabla u|^{p-2}\nabla u'\cdot\nabla\psi+(p-2)|\nabla u|^{p-4}(\nabla u\cdot\nabla u')(\nabla u\cdot\nabla\psi)\Big)\,dx\\ &= (p-1)\int_{\Omega^c}|\nabla u|^{p-2}\nabla u\cdot\nabla u'\,dx=-(p-1)\int_{\partial\Omega}|\nabla u|^{p-2}(\nabla u\cdot n_\Omega)u'\,d\mathcal{H}^1\\ &= (p-1)\int_{\partial\Omega}|\nabla u|^{p-2}(\nabla u\cdot n_\Omega)(V\cdot\nabla u)\,d\mathcal{H}^1,\end{aligned}$$

which is precisely (2.2), since $\nabla u=|\nabla u|\,n_\Omega$ on $\partial\Omega$. Actually let us recall from [21] that the gradient $\nabla u_\Omega(y)$ has (non-tangential) limits as $y\to x\in\partial\Omega$ for $\mathcal{H}^1$-a.e. $x\in\partial\Omega$ (see also [9, Lemma 2.13]), and that, denoting such limits by $\nabla u_\Omega(x)$, we have $|\nabla u_\Omega|\in L^p(\partial\Omega,d\mathcal{H}^1)$.

□

Lemma 2.6 (Shape derivatives with respect to side rotations)**.** *Let F be as in Definition* 2.1. *Let $T\in\mathcal{T}$ and let $\{T_t\}$ be as in Definition* 2.2. *Then*

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}F(T_t) &= (\operatorname{sign}\alpha)\,(p-1)\left(\int_{A_1}^{M}|x-M|\,|\nabla u_T|^p\,d\mathcal{H}^1(x)-\int_{M}^{A_2}|x-M|\,|\nabla u_T|^p\,d\mathcal{H}^1(x)\right)\\ &= (\operatorname{sign}\alpha)\,(p-1)\int_{A_1}^{A_2}\left(\frac{\ell_1}{2}-|x-A_1|\right)|\nabla u_T|^p\,d\mathcal{H}^1(x),\end{aligned}\tag{2.5}$$

$$\frac{d}{dt}\Big|_{t=0}\mathrm{Vol}\,(T_t)=0,\tag{2.6}$$

$$\frac{d}{dt}\Big|_{t=0}\mathrm{Per}\,(T_t)=\frac{\ell_1}{2}(f(\theta_1)-f(\theta_2)),\quad \textit{where}\quad f(\theta)=\cot\theta+\frac{1}{\sin\theta}.\tag{2.7}$$

Proof. If $\Phi_t:\mathbb{R}^2\to\mathbb{R}^2$ is a one-parameter family of diffeomorphisms such that $\Phi_t(T)=T_t$, then by an elementary geometic argument we see that for every $x\in[A_1,A_2]$ such that

$$|x-M|<\frac{\ell_1}{2}\min\left\{1,\left(1+\frac{\sin t\,\cos\theta_1}{\sin(\theta_1-t)}\right),\left(1-\frac{\sin t\,\cos\theta_2}{\sin(\theta_2+t)}\right)\right\},$$

we have

$$\Phi_t(x)\cdot n_T(x)=\begin{cases}|x-M|\,\dfrac{\sin t\,\sin\theta_1}{\sin(\theta_1-t)}\left(1+\dfrac{\sin t\,\cos\theta_1}{\sin(\theta_1-t)}\right)^{-1} & \text{if } x\in[A_1,M],\\ -|x-M|\,\dfrac{\sin t\,\sin\theta_2}{\sin(\theta_2+t)}\left(1-\dfrac{\sin t\,\cos\theta_2}{\sin(\theta_2+t)}\right)^{-1} & \text{if } x\in[M,A_2].\end{cases}$$

Then $V(x):=\dfrac{d}{dt}\Big|_{t=0}\Phi_t(x)$ satisfies

$$V(x)\cdot n_T(x)=\begin{cases}|x-M| & \text{if } x\in[A_1,M],\\ -|x-M| & \text{if } x\in[M,A_2].\end{cases}$$

Now, applying Lemma 2.5 to this family Φ_t, we get (2.5).

On the other hand, through elementary geometric arguments, we see that

$$\begin{aligned}\mathrm{Per}\,(T_t) &= \mathrm{Per}\,(T)-\ell_1+\frac{\ell_1\,\sin\theta_1}{2\,\sin(\theta_1-t)}+\frac{\ell_1\,\sin\theta_2}{2\,\sin(\theta_2+t)}+\frac{\ell_1\,\sin t}{2\,\sin(\theta_1-t)}-\frac{\ell_1\,\sin t}{2\,\sin(\theta_2+t)}\\ \mathrm{Vol}\,(T_t) &= \mathrm{Vol}\,(T)+o(t),\end{aligned}$$

which immediately give (2.6) and (2.7). □

Lemma 2.7 (Shape derivatives with respect to parallel movements)**.** *Let F be as in Definition 2.1. Let $T \in \mathcal{T}$ and let $\{T_t\}$ be as in Definition 2.3. Then*

$$\frac{d}{dt}\Big|_{t=0} F(T_t) = (\operatorname{sign}\alpha)\,(p-1)\int_{A_1}^{A_2} |\nabla u|^p \, d\mathcal{H}^1. \tag{2.8}$$

$$\frac{d}{dt}\Big|_{t=0} \operatorname{Vol}(T_t) = \ell_1. \tag{2.9}$$

$$\frac{d}{dt}\Big|_{t=0} \operatorname{Per}(T_t) = f(\theta_1) + f(\theta_2). \tag{2.10}$$

Proof. Notice that the perturbation described in Definition 2.3 corresponds to the diffeomorphism $\Phi_t(x) = \frac{h+t}{h}x$, where without loss of generality we suppose $A_3 = 0$ and we set h to be the distance from A_3 to the side $[A_1, A_2]$. Thus $V(x) = x/h$ and $V(x)\cdot n_T(x) \equiv 1$ on $[A_1, A_2]$, so we get (2.8), while (2.9) and (2.10) follow directly by the equalities

$$\operatorname{Per}(T_t) = \operatorname{Per}(T) + t\big(f(\theta_1)+f(\theta_2)\big) \qquad \text{and} \qquad \operatorname{Vol}(T_t) = \operatorname{Vol}(T) + t\ell_1 + o(t). \qquad \square$$

3. Proof of Theorems 1.1 and 1.2

We will prove Theorems 1.1 and 1.2 by a reflection argument. We first give the geometric construction, which is the same for the cases of interior or exterior domains, and then we will treat them separately.

Geometric construction. Suppose by contradiction that T has two different inner angles $\theta_1 < \theta_2$. Let M be the midpoint of the side $[A_1, A_2]$ and let $\mathcal{L}_M$ be the line passing through M and orthogonal to the side $[A_1, A_2]$. Without loss of generality we can suppose that $M = 0$, $\mathcal{L}_M$ is the y-axis $\{x = 0\}$, $A_1 = (-\ell_1/2, 0)$, $A_2 = (\ell_1/2, 0)$ and $A_3 = (x_3, y_3)$ with $x_3 > 0$ and $y_3 > 0$ (see Figure 2). Let N be the intersection of $\mathcal{L}_M$ with the side $[A_1, A_3]$. We denote by Ω_{int} the interior of the triangle with vertices M, N and A_2 and by Ω_{ext} the unbounded domain $\{x > 0\} \setminus \overline{T}$. For any function $u : \mathbb{R}^2 \to \mathbb{R}$ we consider the reflection $\tilde{u} : \mathbb{R}^2 \to \mathbb{R}$ of u defined by $\tilde{u}(x, y) = u(-x, y)$.

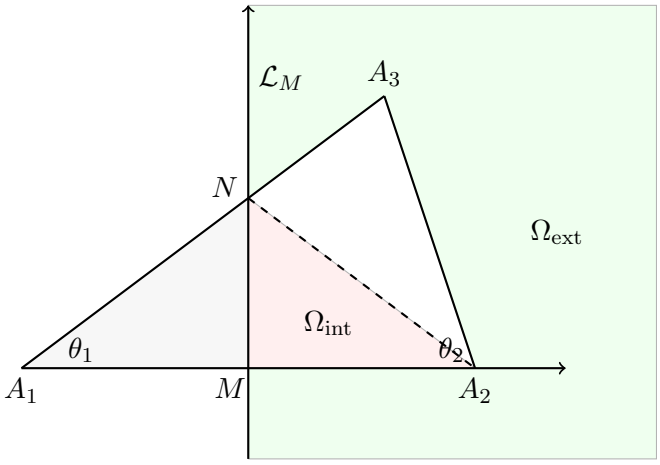

Figure 2. Construction of the reflected domain

Proof of Theorem 1.1. We carry out the argument for the torsion function, namely the solution u_T to (1.1), being the case of the first eigenfunction of the Dirichlet Laplacian completely analogous. By elliptic regularity, u_T is $C^{1,\alpha}$ up to the boundary of T (except at the vertices), and by the strong maximum principle we have that $u_T > 0$ in T (see for instance [17]). We now consider the reflection $\tilde{u}_T$ of u_T. We notice that $\Delta(u_T - \tilde{u}_T) = 0$ in $\Omega_{\rm int}$, while on the boundary $\partial\Omega_{\rm int}$ we have:

$$\tilde{u}_T = u_T \quad \text{on} \quad [N, M] \cup [M, A_2] \qquad \text{and} \qquad 0 = \tilde{u}_T < u_T \quad \text{on} \quad [N, A_2].$$

Thus, $u_T - \tilde{u}_T > 0$ in $\Omega_{\rm int}$ and by Hopf's boundary point lemma

$$|\nabla u_T| > |\nabla \tilde{u}_T| \quad \text{on} \quad [M, A_2]. \tag{3.1}$$

Multiplying this inequality by $|x - M|$ and integrating on the segment $[M, A_2]$, we get

$$\int_M^{A_2} |x - M|\, |\nabla u_T|^2\, d\mathcal{H}^1(x) > \int_M^{A_2} |x - M|\, |\nabla \tilde{u}_T|^2\, d\mathcal{H}^1(x) = \int_{A_1}^{M} |x - M|\, |\nabla u_T|^2\, d\mathcal{H}^1(x)\,, \tag{3.2}$$

which implies

$$\int_{A_1}^{A_2} \left(\frac{\ell_1}{2} - |x - A_1| \right) |\nabla u_T|^2\, d\mathcal{H}^1(x) < 0\,. \tag{3.3}$$

This is clearly in contradiction with (1.2), and also with (1.3) since $\theta_1 < \theta_2$ and the function $\theta \mapsto f(\theta)$ is monotone decreasing on $(0, \pi)$. □

Proof of Theorem 1.2. Let now $u_T : \mathbb{R}^2 \setminus T \to \mathbb{R}$ be the p-capacitary potential of T. By a reflection argument with respect to each side, we get that u_T is $C^{1,\alpha}$ up to the boundary of T (except at the vertices). By the strong maximum principle for p-harmonic functions, we have the strict inequality $u_T < 1$ on the open set $\Omega_{\rm ext} = \mathbb{R}^2 \setminus \overline{T}$ (see [22]). We notice that $\Delta_p \tilde{u}_T = 0 = \Delta_p u_T$ in $\Omega_{\rm ext}$, while on the boundary $\partial\Omega_{\rm ext}$ we have:

$$\tilde{u}_T = u_T \quad \text{on} \quad \mathcal{L}_M \cup [M, A_2] \qquad \text{and} \qquad \tilde{u}_T < u_T = 1 \quad \text{on} \quad [N, A_3] \cup [A_2, A_3].$$

By the strong comparison principle for p-harmonic functions (see [22]) we get that $\tilde{u}_T < u_T$ in $\Omega_{\rm ext}$. As a consequence, by Hopf's principle for p-harmonic functions (see for instance [23, Chapter 5, Section 5.5]) we obtain

$$|\nabla \tilde{u}_T| > |\nabla u_T| \quad \text{on} \quad [M, A_2]. \tag{3.4}$$

Then, similarly as in the case of interior problems, integrating this inequality we get

$$\int_{A_1}^{M} |x - M|\, |\nabla u_T|^p\, d\mathcal{H}^1(x) = \int_M^{A_2} |x - M|\, |\nabla \tilde{u}_T|^p\, d\mathcal{H}^1(x) > \int_M^{A_2} |x - M|\, |\nabla u_T|^p\, d\mathcal{H}^1(x), \tag{3.5}$$

which implies

$$\int_{A_1}^{A_2} \left(\frac{\ell_1}{2} - |x - A_1| \right) |\nabla u_T|^p\, d\mathcal{H}^1(x) > 0. \tag{3.6}$$

This contradicts (1.2). □

4. Proof of Theorem 1.6

Let $T \in \mathcal{T}$. We observe that, if $\{T_t\}$ is the one-parameter family of triangles given by Definition 2.3, for every t the triangle T_t is homothetic to T. Then, since the functionals $\mathcal{E}_{\rm Vol}(T)$ and $\mathcal{E}_{\rm Per}(T)$ from (2.1) are invariant under dilations, it holds

$$\frac{d}{dt}\Big|_{t=0} \mathcal{E}_{\rm Vol}(T_t) = 0 \qquad \text{and} \qquad \frac{d}{dt}\Big|_{t=0} \mathcal{E}_{\rm Per}(T_t) = 0\,. \tag{4.1}$$

The derivatives appearing in the above equations can be explicitly computed by using formulas (2.8)-(2.9)-(2.10) in Lemma 2.6. This way we see that the two equalities (4.1) correspond exactly to (1.7) and (1.8), the value of κ being given as in the statement of Theorem 1.6. Since T is an arbitrary triangle, our proof is achieved. □

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