Large Deviation Principle for invariant distributions of Memory Gradient Diffusions

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Abstract

In this paper, we consider a class of diffusion processes based on a memory gradient descent, *i.e.* whose drift term is built as the average all along the past of the trajectory of the gradient of a coercive function U. Under some classical assumptions on U, this type of diffusion is ergodic and admits a unique invariant distribution. In view to optimization applications, we want to understand the behaviour of the invariant distribution when the diffusion coefficient goes to 0. In the non-memory case, the invariant distribution is explicit and the so-called Laplace method shows that a Large Deviation Principle (LDP) holds with an explicit rate function, that leads to a concentration of the invariant distribution around the global minima of U. Here, except in the linear case, we have no closed formula for the invariant distribution but we show that a LDP can still be obtained. Then, in the one-dimensional case, we get some bounds for the rate function that lead to the concentration around the global minimum under some assumptions on the second derivative of U.

Keywords: Large Deviation Principle, Hamilton-Jacobi Equations, Freidlin and Wentzell Theory, small stochastic perturbations, hypoelliptic diffusions

1 Introduction

The aim of this paper is to study some asymptotic properties of a diffusive stochastic model with memory gradient when the noise component vanishes. The evolution is given by the following stochastic differential equation (SDE)on $\mathbb{R}^d$:

$$dX_t^\varepsilon = \varepsilon dB_t - \left(\frac{1}{k(t)}\int_0^t k'(s)\nabla U(X_s^\varepsilon)ds\right)dt, \tag{1.1}$$

where $\varepsilon > 0$ and (B_t) is a standard d-dimensional Brownian motion. A special feature of such equation is the integration over the past of the trajectory depending on a function k which quantifies the amount of memory. Our work is mainly motivated by optimization considerations. Indeed, in a recent work Cabot, Engler, and Gadat (2009b) have shown that the solution of the deterministic dynamical system ($\varepsilon = 0$) converges to the minima of the potential U. Without memory, that is without integration over the past of the trajectory, the model (1.1) reduces to the classical gradient descent model and such convergence results are well-known. Even in the deterministic framework, a potential interest of the gradient with memory is the capacity of the solution to avoid some local traps of U. Indeed, the solution of (1.1) (when $\varepsilon = 0$) may keep some inertia even when it reaches a local minimum of U which implies a larger exploration of the space than a classical gradient descent which cannot escape from local minima (see Alvarez (2000) and Cabot (2009)). Usually such property is obtained by introducing a small noise term. In the classical case, this leads to the usual following SDE

$$dX_t^\varepsilon = \varepsilon dB_t - \nabla U(X_t^\varepsilon)dt. \tag{1.2}$$

As mentioned above, the behaviour of the invariant distribution of this model when ε goes to 0 is well-known. Using the so-called Laplace method, it can be proved that a Large Deviation Principle (LDP) holds and that the invariant distribution of (1.2) concentrates on the global minima of U when the parameter $\varepsilon \to 0$ (see *e.g.* Freidlin and Wentzell (1979)).

It is then natural to investigate the study of the stochastic memory gradient (1.1) in order to obtain similar results. A major difference with the usual gradient diffusion is that the integration over the past of the trajectory makes the process $(X_t^\varepsilon)_{t\geq 0}$ non Markov. This can be overcome with the introduction of an auxiliary (Y_t^ε) defined by

$$Y_t^\varepsilon = \left(\frac{1}{k(t)}\int_0^t k'(s)\nabla U(X_s^\varepsilon)ds\right)dt. \tag{1.3}$$

In general, the couple $Z_t^\varepsilon = (X_t^\varepsilon, Y_t^\varepsilon)$ gives rise to a non-homogeneous Markov process (see Gadat and Panloup (2012)). In order to consider the notion of invariant measure, we concentrate on the case where $k(t) = e^{\lambda t}$ which turns (Z_t^ε) into an homogeneous Markov process. In this context, Gadat and Panloup (2012) have shown existence and uniqueness of the invariant measure ν_ε for (Z_t^ε).

In the present work, our objective is to obtain some sharp estimations of the asymptotic behaviour of (ν_ε) as $\varepsilon \to 0$. More precisely, we shall first show that $(\nu_\varepsilon)_{\varepsilon>0}$ satisfies a Large Deviation Principle. Then, we will try to obtain some sharp bounds for the associated rate function in order to understand how the invariant probability is distributed as $\varepsilon \to 0$. In particular, we will establish the concentration around the global minima of U up to technical hypotheses. In the classical setting of (1.2), this is an essential step towards implementing the strategy of the so-called *simulated annealing*. Developing a simulated annealing optimization procedure to the memory gradient diffusion is certainly a motivation of the study of (1.1). This will be addressed in a forthcoming work.

The paper is also motivated by extending some results of Large Deviations for invariant distributions to a difficult context where the process is not elliptic and the drift vector field is not the gradient of a potential. These two points and especially the second one strongly complicate the problem since explicit computations of the invariant measure are generally impossible. This implies that the works on elliptic Kolmogorov equations by Chiang, Hwang, and Sheu (1987), Miclo (1992) or Holley and Stroock (1988) for instance, can not be extended to our context. For similar considerations in other non-Markov models, one should also mention the recent works on Mac-Kean Vlasov diffusions by Herrmann and Tugaut (2010) and on self-interacting (with attractive potential) diffusions by Raimond (2009).

Here, in order to obtain a LDP for $(\nu_\varepsilon)_{\varepsilon\geq 0}$ we adapt the strategy of Puhalskii (2003) and Freidlin and Wentzell (1979) to our degenerated context. We shall first show a finite-time LDP for the underlying stochastic process. Second we prove the exponential tightness of $(\nu_\varepsilon)_{\varepsilon\geq 0}$ by using Lyapunov type arguments. Finally, we show that the associated rate function, denoted as W in the paper, can be expressed as the solution of a control problem (in an equivalent way as the solution of a Hamilton-Jacobi equation). However, at first sight the solution of the control problem is not unique. This uniqueness property follows from an adaptation of the results of Freidlin and Wentzell (1979) to our framework. In particular, we obtain a formulation of the rate function in terms of the costs to join stable critical points of our dynamical system. Next, the second step of the paper (sharp estimates of W) is investigated by the study of the cost to join stable critical points.

The paper is then organized as follows. In Section 2, we recall some results about the long-time behaviour of the diffusion when ε is fixed. Moreover, we provide the main assumptions needed for obtaining the LDP for (ν_ε). In Section 3, we prove the exponential tightness of (ν_ε) and show that any rate function W associated with a convergent subsequence is a solution of a finite or infinite time control problem. In Section 4, we prove the uniqueness of W by adapting the Freidlin and Wentzell approach to our context (see also the works of (Biswas & Budhiraja, 2011) and (Cerrai & Röckner, 2005) for other adaptations of this theory). Since the study of the cost function is quite hard in a general setting, we focus in Section 5 on the case of a double-well potential U. In this context, we obtain some upper and lower bounds for the associated quasi-potential function. Then, we provide some conditions on U and on the memory parameter λ which allow us to prove the concentration of the invariant distribution around the global minima. Note that, even if our assumptions in this part seem a little bit restrictive, the proofs of the bounds (especially the lower bound) are obtained by an original (and almost optimal) use of some Lyapunov functions associated with the dynamical system.

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2 Setting and Main Results

2.1 Notations and background on Large Deviation theory

In the paper, the scalar product and the Euclidean norm on $\mathbb{R}^d$ are respectively denoted by $\langle\, ,\, \rangle$ and $|\,.\,|$. The space of $d \times d$ real-valued matrices is referred as $\mathbb{M}_d(\mathbb{R})$ and we use the notation $\|\,.\,\|$ for the Frobenius norm on $\mathbb{M}_d(\mathbb{R})$.

We denote by $\mathbb{H}(\mathbb{R}_+, \mathbb{R}^d)$ the Cameron-Martin space, *i.e.* the set of absolutely continuous functions $\varphi : \mathbb{R}_+ \to \mathbb{R}^d$ such that $\varphi(0) = 0$ and such that $\dot{\varphi} \in L^{2,\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^d)$.

For a $\mathcal{C}^2$-function $f : \mathbb{R}^d \to \mathbb{R}$, ∇f and $D^2 f$ denote respectively the gradient of f and the Hessian matrix of f. In the one-dimensional case, we will switch to the notation f' and f" in order to emphasize the difference with $d > 1$. Given any $f \in \mathcal{C}^2(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})$, $\nabla_x f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and $D_x^2 f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{M}_d(\mathbb{R})$ denote the functions respectively defined by $(\nabla_x f(x,y))_i = \partial_{x_i} f(x,y)$ and $(D_x^2 f(x,y))_{i,j} = \partial_{x_i}\partial_{x_j} f(x,y)$. Obviously these notations are naturally extended to $\nabla_y f$, $D^2_{x,y} f$ and $D^2_y f$. Finally, for any vector $v \in \mathbb{R}^d$, v^t will refer to the transpose of v.

For a measure μ and a μ-measurable function f, we set $\mu(f) = \int f d\mu$.

Let us now recall some definitions relative to the Large Deviation theory (see Dembo and Zeitouni (2010) for further references on the subject). Let (E, d) denote a metric space. A family of probability measures $(\nu_\varepsilon)_{\varepsilon>0}$ on E satisfies a Large Deviation Principle (shortened as LDP) with speed r_ε and rate function I if for all open set O and closed set F,

$$\liminf_{\varepsilon\to 0} r_\varepsilon \log(\nu_\varepsilon(O)) \geq -\inf_{x\in O} I(x) \quad \text{and} \quad \limsup_{\varepsilon\to 0} r_\varepsilon \log(\nu_\varepsilon(F)) \leq -\inf_{x\in F} I(x).$$

The function I is referred to be *good* if for any $c \in \mathbb{R}$, $\{x \in E, I(x) \leq c\}$ is compact. In this paper, we will use some classical compactness results in Large Deviation theory. A family of probability measures $(\nu_\varepsilon)_{\varepsilon>0}$ is said to be *exponentially tight of order* r_ε if

$$\forall a > 0,\ \exists K_a \text{ compact of } E \text{ such that } \limsup_{\varepsilon\to 0} r_\varepsilon \log(\nu_\varepsilon(K_a^c)) \leq -a.$$

Then, we recall the link between exponential tightness and the Large Deviation Principle (see Feng and Kurtz (2006), chapter 3 for instance).

Proposition 2.1 *Let (S, d) be a metric space and $(\nu_\varepsilon)_{\varepsilon\geq 0}$ a sequence of exponentially tight probability measures on the Borel σ-algebra of S with speed r_ε. Then there exists a subsequence $(\varepsilon_k)_{k\geq 0}$ such that $\varepsilon_k \to 0$ along which the LDP holds with good rate function I and speed r_{ε_k}.*

Definition 2.1 *Such subsequence $(\nu_{\varepsilon_k})_{k\geq 1}$ will be called a (LD)-convergent subsequence.*

2.2 Averaged gradient diffusions

Throughout this paper, we denote by $U : \mathbb{R}^d \mapsto \mathbb{R}$ a smooth (at least $\mathcal{C}^2$) function on $\mathbb{R}^d$ and coercive, *i.e.*

$$\inf_{x\in\mathbb{R}} U(x) > 0, \qquad \lim_{|x|\to+\infty} U(x) = +\infty, \quad \text{and} \quad \liminf_{|x|\to+\infty} \langle x, \nabla U(x)\rangle > 0. \tag{2.1}$$

As announced in Introduction with $k(t) = e^{\lambda t}$, we are interested in the stochastic evolution of

$$dX_t^\varepsilon = \varepsilon dB_t - \left(\lambda e^{-\lambda t}\int_0^t e^{\lambda s}\nabla U(X_s^\varepsilon)ds\right)dt,$$

where $\lambda > 0$ and $(B_t)_{t\geq 0}$ is a standard d-dimensional Brownian motion. The process $(X_t^\varepsilon)_{t\geq 0}$ is not a Markov process but enlarging the space by defining the auxiliary process $(Y_t^\varepsilon)_{t\geq 0}$ as

$$Y_t^\varepsilon = \lambda e^{-\lambda t}\int_0^t e^{\lambda s}\nabla U(X_s^\varepsilon)ds,$$

then $(Z_t^\varepsilon)_{t\geq 0} := ((X_t^\varepsilon, Y_t^\varepsilon))_{t\geq 0}$ is a Markov process (see Gadat and Panloup (2012) for instance). More precisely $(Z_t^\varepsilon)_{t\geq 0}$ satisfies:

$$\begin{cases} dX_t^\varepsilon = \varepsilon dB_t - Y_t^\varepsilon dt, \\ dY_t^\varepsilon = \lambda(\nabla U(X_t^\varepsilon) - Y_t^\varepsilon)dt. \end{cases} \tag{2.2}$$

When necessary, we will denote by $(Z_t^{\varepsilon,z})_{t\geq 0}$ the solution starting from $z \in \mathbb{R}^d$ and by $\mathbb{P}_z^\varepsilon$ the distribution of this process on $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$. In the sequel, we will also intensively use the deterministic system obtained when $\varepsilon = 0$ in (2.2): if $(z(t))_{t\geq 0} := (x(t), y(t))_{t\geq 0}$, the canonical differential system is

$$\dot{z}(t) = b(z(t)) \quad \text{with} \quad b(x,y) = \begin{pmatrix} 0 & -y \\ \lambda\nabla U(x) & -\lambda y \end{pmatrix}. \tag{2.3}$$

2.3 Assumptions

The function ∇U being not necessarily Lipschitz continuous, we assume in all the paper that there exists $C > 0$ such that for all $x \in \mathbb{R}^d$, $\|D^2U(x)\| \leq CU(x)$. This assumption ensures the non-explosion (in finite horizon) of $(Z_t^\varepsilon)_{t\geq 0}$ (see Proposition 2.1 of Gadat and Panloup (2012)). Since ∇U is locally Lipschitz continuous, existence and uniqueness hold for the solution of (2.2) and $(Z_t^\varepsilon)_{t\geq 0}$ is a Markov process with semi-group denoted by $(P_t^\varepsilon)_{t\geq 0}$ and infinitesimal generator $\mathcal{A}^\varepsilon$ defined, for all $f \in \mathcal{C}_c^2(\mathbb{R}^d \times \mathbb{R}^d)$, by:

$$\mathcal{A}^\varepsilon f(x,y) = -\langle y, \partial_x f\rangle + \lambda\langle \nabla U(x) - y, \partial_y f\rangle + \frac{\varepsilon^2}{2}\mathrm{Tr}\left(D_x^2 f\right). \tag{2.4}$$

We first recall some results obtained by Gadat and Panloup (2012) on existence and uniqueness for the invariant distribution of (2.2). To this end, we need to introduce a mean-reverting assumption denoted by $(\mathbf{H_{mr}})$ and some hypoellipticity assumption $(\mathbf{H_{Hypo}})$. The mean reverting assumption is expressed as follows:

$(\mathbf{H_{mr}})$: $\quad \lim_{|x|\to+\infty}\langle x, \nabla U(x)\rangle = +\infty \quad$ and $\quad \lim_{|x|\to+\infty} \frac{\|D^2U(x)\|}{\langle x, \nabla U(x)\rangle} = 0.$

Concerning the second assumption, let us define $\mathcal{E}_U$ by

$$\mathcal{E}_U = \left\{x \in \mathbb{R}^d,\ \det\left(D^2U(x)\right) \neq 0\right\}, \tag{2.5}$$

and denote by $\mathcal{M}_U$ the complementary manifold $\mathcal{M}_U = \mathbb{R}^d \setminus \mathcal{E}_U$. Assumption $(\mathbf{H_{Hypo}})$ is then defined by:

$(\mathbf{H_{Hypo}})$: U is $\mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R})$, $\quad \lim_{|x|\to+\infty} \frac{U(x)}{|x|} = +\infty \quad$ and $\quad \dim(\mathcal{M}_U) \leq d-1$.

The above assumption implies the uniqueness invariant distribution: the smoothness of U and the fact that $\dim(\mathcal{M}_U) \leq d-1$ ensure that the Hörmander condition is satisfied on a sufficiently large subspace of $\mathbb{R}^{2d}$ whereas the fact that $\lim |x|^{-1}U(x) = +\infty$ as $|x| \to +\infty$ (which implies that U grows at least linearly) is needed for the topological irreducibility of the semi-group (see Gadat and Panloup (2012) for details). Under these assumptions, we deduce the following proposition from Theorems 2.3 and 3.2 of Gadat and Panloup (2012):

Proposition 2.2 *Assume* $(\mathbf{H_{mr}})$*. Then, for all* $\varepsilon > 0$*, the solution of* (2.2) *admits an invariant distribution. Furthermore, if* $(\mathbf{H_{Hypo}})$ *holds, the invariant distribution is unique and admits a* λ_{2d}*-a.s. positive density. We denote by* ν_ε *this invariant distribution.*

Note that $(\mathbf{H_{mr}})$ implies Assumption $(\mathbf{H_1})$ of Gadat and Panloup (2012) in the particular case $\sigma = I_d$ and $r_\infty = \lambda$.

Our goal is now to obtain a Large Deviation Principle for $(\nu_\varepsilon)_{\varepsilon>0}$ when $\varepsilon \to 0$. To this end, we need to introduce some more constraining mean-reverting assumptions than $(\mathbf{H_{mr}})$:

$(\mathbf{H_{Q+}})$: There exists $\rho \in (0,1)$, $C > 0$, $\beta \in \mathbb{R}$ and $\alpha > 0$ such that

$$\begin{array}{lll} (i) & -\langle x, \nabla U(x)\rangle \leq \beta - \alpha U(x), \forall x \in \mathbb{R}^d & \\ (ii) & |\nabla U|^2 \leq C(1 + U^{2(1-\rho)}) \quad \text{and} & \displaystyle\lim_{|x|\to+\infty} \frac{\|D^2U(x)\|}{U(x)} = 0. \end{array}$$

$(\mathbf{H_{Q-}})$: There exists $a \in (1/2, 1]$, $C > 0$, $\beta \in \mathbb{R}$ and $\alpha > 0$ such that

$$\begin{array}{ll} (i) & -\langle x, \nabla U(x)\rangle \leq \beta - \alpha |x|^{2a}, \forall x \in \mathbb{R}^d \\ (ii) & |\nabla U|^2 \leq C(1+U) \quad \text{and} \quad \displaystyle\sup_{x\in\mathbb{R}^d} \|D^2U(x)\| < +\infty. \end{array}$$

Remark 2.1 *Assumptions* $(\mathbf{H_{Q+}})$ *and* $(\mathbf{H_{Q-}})$ *correspond respectively to super-quadratic and subquadratic potentials. For instance, assume that* $U(x) = (1+|x|^2)^p$*. When* $p \geq 1$*,* $(\mathbf{H_{Q+}})$ *holds with* $\rho \in (0, \frac{1}{2p})$ *and if* $p \in (1/2, 1]$*,* $(\mathbf{H_{Q-}})$ *holds with* $a = p$*. These assumptions are adapted to a large class of potentials* U *with polynomial growth (more than linear). However, they do not cover the potentials with exponential growth ((*$\mathbf{H_{Q+}}$*)*(ii) *is no longer fulfilled).*

2.4 Main results

2.4.1 Exponential tightness and Hamilton Jacobi equation

Let $\varphi \in \mathbb{H}$. When existence holds, we denote respectively by $\mathbf{z}_\varphi := (\mathbf{z}_\varphi(t))_{t\geq0}$ and by $\tilde{\mathbf{z}}_\varphi := (\tilde{\mathbf{z}}_\varphi(t))_{t\geq0}$, a solution of

$$\dot{\mathbf{z}}_\varphi = b(\mathbf{z}_\varphi) + \begin{pmatrix} \dot{\varphi} \\ 0 \end{pmatrix} \quad \text{and} \quad \dot{\tilde{\mathbf{z}}}_\varphi = -b(\tilde{\mathbf{z}}_\varphi) + \begin{pmatrix} \dot{\varphi} \\ 0 \end{pmatrix}. \tag{2.6}$$

Note that $(\mathbf{H_{Q+}})$ and $(\mathbf{H_{Q-}})$ ensure the finite-time non-explosion of $\mathbf{z}_\varphi$ and $\tilde{\mathbf{z}}_\varphi$ for all $\varphi \in \mathbb{H}$ (see *e.g.* Equation (3.4)). Thus, since ∇U is locally Lipschitz continuous, for all $z \in \mathbb{R}^{2d}$, the solutions starting from z respectively denoted by $\mathbf{z}_\varphi(z,.)$ and $\tilde{\mathbf{z}}_\varphi(z,.)$ exist and are unique.

Finally, we will also need the following assumption.

$(\mathbf{H_D})$: The set of critical points $(x_i^\star)_{i=1\ldots\ell}$ of U is finite and each $D^2U(x_i^\star)$ is invertible.

This assumption will be necessary to obtain some uniqueness property. We are now able to express our first main result.

Theorem 2.1 *Assume that* $(\mathbf{H_{Hypo}})$ *holds and that either* $(\mathbf{H_{Q+}})$ *or* $(\mathbf{H_{Q-}})$ *is satisfied. Then,*

(i) The family $(\nu_\varepsilon)_{\varepsilon\in(0,1]}$ *is exponentially tight on* $\mathbb{R}^{2d}$ *with speed* ε^{-2}.

(ii) Let $(\nu_{\varepsilon_n})_{n\geq 1}$ *be a (LD)-convergent subsequence and denote by* W *the associated (good) rate function. Then,* W *satisfies for all* $t \geq 0$ *and any* $z \in \mathbb{R}^d \times \mathbb{R}^d$:

$$W(z) = \inf_{\varphi\in\mathbb{H}} \left[\frac{1}{2}\int_0^t |\dot{\varphi}(s)|^2 ds + W(\tilde{\mathbf{z}}_\varphi(z,t))\right]. \tag{2.7}$$

(iii) Furthermore, assume that $(\mathbf{H_D})$ *is fulfilled, then*

$$W(z) = \min_{1\leq i\leq \ell} \inf_{\begin{cases}\varphi\in\mathbb{H}\\ \tilde{\mathbf{z}}_\varphi(z,+\infty) = z_i^\star\end{cases}} \left[\frac{1}{2}\int_0^\infty |\dot{\varphi}(s)|^2 ds + W(z_i^\star)\right]. \tag{2.8}$$

where $\tilde{\mathbf{z}}_\varphi(z,+\infty) := \lim_{t\to+\infty}\tilde{\mathbf{z}}_\varphi(z,t)$ *(when exists) and* $z_i^\star = (x_i^\star, 0)$ *for all* $i = 1,\ldots,l$.

Equation (2.7) satisfied by W may be seen as an Hamilton-Jacobi equation (see *e.g.* Barles (1994) for further details on such equations).

2.4.2 Freidlin and Wentzell estimates

Let us stress that the main problem in the expression (2.8) is that the uniqueness is only available conditionally to the values of $W(z_i^\star)$, $i = 1,\ldots\ell$. Thus, in order to obtain a LDP, we now need to show that that this uniqueness is not conditional, *i.e.* that the values of $W(z_i^\star)$ are uniquely determined. We are going to obtain this result by following the Freidlin and Wentzell (1979) approach. To this end, we first recall some useful elements of Freidlin and Wentzell theory.

$\{i\}$-Graphs Following the notations of Theorem 2.1, we denote by $\{z_1^\star,\ldots,z_\ell^\star\}$ this finite set of equilibria and we recall here the definition of $\{i\}$-Graphs defined on this set. For any $i\in\{1,\ldots,\ell\}$, we denote by $\mathcal{G}(i)$ the set of oriented graphs with vertices $\{z_1^\star,\ldots,z_\ell^\star\}$ that satisfy the three following properties.

(i) Each state $z_j^\star \neq z_i^\star$ is the initial point of exactly one oriented edge in the graph.
(ii) The graph does not have any cycle.
(iii) For any $z_j^\star \neq z_i^\star$, there exists a (unique) path composed of oriented edges starting at state $z_j^\star$ and leading to the state $z_i^\star$.

L^2 control cost between between equilibria We now define for any couple of points $(\xi_1,\xi_2)\in(\mathbb{R}^d\times\mathbb{R}^d)^2$ the minimal L^2 cost to go from ξ_1 to ξ_2 within a finite time t as

$$I_t(\xi_1,\xi_2) = \inf_{\begin{cases}\varphi\in\mathbb{H}\\ \mathbf{z}_\varphi(\xi_1,t) = \xi_2\end{cases}} \frac{1}{2}\int_0^t |\dot{\varphi}(s)|^2 ds,$$

and also the minimal L^2 cost to go from ξ_1 to ξ_2 within any time:

$$I(\xi_1,\xi_2) = \inf_{t\geq 0} I_t(\xi_1,\xi_2).$$

The function I is usually called the *quasipotential*. With these definitions, one can obtain the Freidlin and Wentzell estimate which gives another representation of $W(z_i^\star)$, $i=1,\ldots,\ell$.

Theorem 2.2 *Assume that* $(\mathbf{H_{Hypo}})$ *holds and that either* $(\mathbf{H_{Q+}})$ *or* $(\mathbf{H_{Q-}})$ *is satisfied. If* $(\mathbf{H_D})$ *holds, then, for any (LD)-convergent subsequence* $(\nu_{\varepsilon_n})_{n\geq 1}$*, the associated rate function* W *satisfies:*

$$\forall i \in \{1\ldots\ell\} \qquad W(z_i^\star) = \mathcal{W}(z_i^\star) - \min_{j\in\{1,\ldots,\ell\}} \mathcal{W}(z_j^\star)$$

where

$$\forall i \in \{1\ldots\ell\} \qquad \mathcal{W}(z_i^\star) := \min_{\mathcal{IG}\in\mathcal{G}(i)} \sum_{(z_m^\star\to z_n^\star)\in\mathcal{IG}} I(z_m^\star, z_n^\star). \tag{2.9}$$

The next corollary follows immediately from Theorem 2.1 and Theorem 2.2.

Corollary 1 *Assume that* $(\mathbf{H_{Hypo}})$ *holds and that either* $(\mathbf{H_{Q+}})$ *or* $(\mathbf{H_{Q-}})$ *is satisfied. If* $(\mathbf{H_D})$ *holds,* (ν_ε) *satisfies a large deviation principle with speed* ε^{-2} *and good rate function* W *such that*

$$W(z) = \min_{1\leq i\leq \ell} \inf_{\begin{cases}\varphi\in\mathbb{H}\\ \tilde{\mathbf{z}}_\varphi(z,+\infty) = z_i^\star\end{cases}} \left[\frac{1}{2}\int_0^\infty |\dot\varphi(s)|^2 ds + \mathcal{W}(z_i^\star)\right] - \min_{j\in\{1,\ldots,\ell\}} \mathcal{W}(z_j^\star),$$

where $\mathcal{W}(z_i^\star)$ *is given by (2.9).*

Case of a double-well non-convex potential In the sequel, we are interested by the location of the minimum of W. More precisely, we expect that this minimum is located on the set of global minima of U. Using Equation (2.8), this point is clear when U is a stricly convex potential. Regarding now the non-convex case, the situation is more involved. Thus, we only focus on the double-well one-dimensional case. Without loss of generality, we assume that U has two local minima denoted by $x_1^\star$ and $x_2^\star$ with

$$x_1^\star < x^\star < x_2^\star \quad \text{and} \quad U(x_1^\star) < U(x_2^\star), \tag{2.10}$$

where $x^\star$ is the unique local maximum between x_1 and x_2. We obtain the following result:

Theorem 2.3 *Assume the hypothesis of Corollary 1 and that* U *satisfies* (2.10)*. Then,*
(i) $\mathcal{W}$ *satisfies*

$$\mathcal{W}(z_1^\star) = I(z_2^\star, z_1^\star) \leq 2[U(x^\star) - U(x_2)].$$

(ii) For all $\alpha\in[0,2]$*, there exists an explicit constant* $m_\lambda(\alpha)$ *such that*

$$\|U''\|_\infty \leq m_\lambda(\alpha) \Longrightarrow \mathcal{W}(z_2^\star) = I(z_1^\star, z_2^\star) \geq \alpha[U(x^\star) - U(x_1)].$$

(iii) As a consequence, if U *satisfies* $\|U''\|_\infty < m_\lambda\left(2\frac{U(x^\star)-U(x_2)}{U(x^\star)-U(x_1)}\right)$*, then*

$$\mathcal{W}(z_1^\star) < \mathcal{W}(z_2^\star),$$

and finally $(\nu_\varepsilon)_{\varepsilon\geq 0}$ *weakly converges towards* $\delta_{z_1^\star}$ *as* $\varepsilon\to 0$.

In the next sections, we prove the above statements. Note that throughout the rest of the paper, C will stand for any non-explicit constant. Note also that except in Section 5, we will prove all the results with $\lambda = 1$ for sake of convenience (one can deduce similar convergences with small modifications for any $\lambda > 0$).

3 Large Deviation Principle for invariant measures $(\nu_\varepsilon)_{\varepsilon\in(0,1]}$

This section describes the proof of Theorem 2.1 which contains two important parts. The first one concerns the exponential tightness of the invariant measures $(\nu_\varepsilon)_{\varepsilon\in(0,1]}$ and the second result is a functional equality for *any* good rate function associated to *any* (LD)-convergent subsequence $(\nu_{\varepsilon_k})_{k\geq 0}$.

We first establish a LDP for $(Z^\varepsilon)_{\varepsilon>0}$ on $\mathcal{C}(\mathbb{R}_+,\mathbb{R}^{2d})$ (space of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}^{2d}$) and then we detail how one can derive the exponential tightness property of $(\nu_\varepsilon)_{\varepsilon\in(0,1]}$ using suitable Lyapunov functions for our dynamical system. Finally, we show that a functional equality such as (2.7) holds.

3.1 Large Deviation Principle for $(Z^\varepsilon)_{\varepsilon>0}$

The next lemma establishes a LDP for trajectories of $((Z^\varepsilon_t)_{t\geq 0})_{\varepsilon>0}$ within a finite time.

Lemma 3.1 *Assume* $(\mathbf{H_{Q+}})$ *or* $(\mathbf{H_{Q-}})$. *Let* $z\in\mathbb{R}^{2d}$ *and* $(z_\varepsilon)_{\varepsilon>0}$ *be a net of* $\mathbb{R}^{2d}$ *such that* $z_\varepsilon \xrightarrow{\varepsilon\to 0} z$. *Then,* $(Z^{\varepsilon,z_\varepsilon})_{\varepsilon>0}$ *satisfies a LDP on* $\mathcal{C}(\mathbb{R}_+,\mathbb{R}^{2d})$ *(endowed with the topology of uniform convergence on compact sets) with speed* ε^{-2}. *The corresponding (good) rate function* $\mathcal{I}_z$ *is defined for all absolutely continuous* $(\mathbf{z}(t))_{t\geq 0}=(\mathbf{x}(t),\mathbf{y}(t))_{t\geq 0}$ *by*

$$\mathcal{I}_z((\mathbf{z}(t))_{t\geq 0})=\inf_{\varphi\in\mathbb{H},\mathbf{z}_\varphi(z,.)=\mathbf{z}(.)}\frac{1}{2}\int_0^\infty|\dot\varphi(s)|^2ds=\frac{1}{2}\int_0^\infty|\dot{\mathbf{x}}(s)+\mathbf{y}(s)|^2\,ds,$$

where $\mathbf{z}_\varphi(z,.)=\mathbf{z}(.)$ *means that for all* $t\geq 0$, $\mathbf{z}_\varphi(z,t)=\mathbf{z}(t)$. *In particular, for all* $t\geq 0$, $(P^\varepsilon_t(z_\varepsilon,.))_{\varepsilon>0}$ *satisfies a LDP with speed* ε^{-2}. *The corresponding rate function* $I_t(z,.)$ *is defined for all* $z,z'\in\mathbb{R}^{2d}$ *by:*

$$I_t(z,z')=\inf_{\mathbf{z}(.)\in\mathcal{Z}_t(z,z')}\mathcal{I}_z(\mathbf{z}(.)),\tag{3.1}$$

where $\mathcal{Z}_t(z,z')$ *denotes the set of absolutely continuous functions* $\mathbf{z}(.)$ *such that* $\mathbf{z}(0)=z$, $\mathbf{z}(t)=z'$. *Furthermore, the function* I_t *can be written as*

$$I_t(z,z')=\inf_{\varphi\in\mathbb{H},\mathbf{z}_\varphi(z,t)=z'}\frac{1}{2}\int_0^t|\dot\varphi(s)|^2ds.$$

Remark 3.1 *Note that such result is quite classical when* $z_\varepsilon=z$ *and when the coefficients are Lipschitz continuous functions (see e.g. Azencott (1980) for instance). Here, we have to handle the possibly super-linear growth of the drift vector field* b *(and also the degeneracy of the diffusion).*

Proof : We wish to apply Theorem 5.2.12 of Puhalskii (2001). To this end, we need to prove the following four points:

• Uniqueness for the *maxingale problem*: this step is an identification of the (potential) *LD*-limits of $(Z^\varepsilon)_{\varepsilon>0}$. More precisely, we need to prove that the *idempotent probability* $\pi_z(.):=\exp(-\mathcal{I}_z(.))$ is the unique solution to the *maxingale problem* (z,G) where $G:\mathbb{R}^{2d}\times\mathcal{C}(\mathbb{R}_+,\mathbb{R}^{2d})\to\mathcal{C}(\mathbb{R}_+,\mathbb{R}^{2d})$ is given by:

$$\forall\,\lambda=(\lambda_1,\lambda_2)\in\mathbb{R}^d\times\mathbb{R}^d,\ \forall\mathbf{z}\in\mathcal{C}(\mathbb{R}_+,\mathbb{R}^{2d}),\ \forall t\geq 0, G_t(\lambda,\mathbf{z})=\int_0^t b(\mathbf{z}(s))ds+\frac{1}{2}\lambda_1^2.$$

The fact that π_z solves the maxingale problem follows from Theorem 3.1 and Lemma 3.2 of (Puhalskii, 2004). Setting $\mathcal{E}(x,y)=U(x)+\dfrac{|y|^2}{2}$, note that Lemma 3.2 can be applied since $\langle\nabla\mathcal{E}(x,y),b(x,y)\rangle\leq 0$ (see condition (3.6a) of (Puhalskii, 2004)). Furthermore, since b is locally Lipschitz continuous, for all $\varphi\in\mathbb{H}$, the ordinary differential equation

$$\dot{\mathbf{z}}=b(\mathbf{z})+\begin{pmatrix}\dot\varphi\\0\end{pmatrix},$$

has a unique solution. Thus, uniqueness for the maxingale problem is a consequence of the second point of Lemma 2.6.17 of (Puhalskii, 2001) and of Theorem 3.1 of (Puhalskii, 2004).

- Continuity condition: some continuity conditions must be satisfied for the characteristics of the diffusion. In fact, since the diffusive component is constant, it is enough to focus on the drift component and to show that for all $t \geq 0$ the function ϕ_t from $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^{2d})$ to $\mathbb{R}^{2d}$ defined by $\phi_t(\mathbf{z}) = \int_0^t b(\mathbf{z}(s))ds$ is a continuous function of $\mathbf{z}$. Since b is Lipschitz continuous on every compact set of $\mathbb{R}^{2d}$, this point is obvious.

- Local majoration condition: in this step, we have to check that for all $M > 0$, there exists an increasing continuous map $\bar{F}^M : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\forall 0 \leq s \leq t \qquad \sup_{\mathbf{z}\in\mathcal{C}(\mathbb{R}_+,\mathbb{R}^{2d}), \|\mathbf{z}\|_\infty \leq M} (\phi_t(\mathbf{z}) - \phi_s(\mathbf{z})) \leq \bar{F}^M(t) - \bar{F}^M(s).$$

with $\|\mathbf{z}\|_\infty = \sup_{t\geq 0} |\mathbf{z}(t)|$. Since b is locally bounded, this point is true with

$$\bar{F}_t^M(t) = \sup_{z\in\mathbb{R}^{2d}, |z|\leq M} |b(z)|t.$$

- Non-Explosion condition (NE): The Non-Explosion condition holds if

  (i) The function π_z defined by $\pi_z := \exp(-\mathcal{I}_z(\mathbf{z}))$ is upper-compact,

  (ii) For all $t \geq 0$ and for all $a \in (0,1]$, the set $\bigcup_{s\leq t} \left\{ \sup_{u\leq s} |\mathbf{z}(u)|,\ \pi_{z,s}(\mathbf{z}) \geq a \right\}$ is bounded where

$$\forall\, z \in \mathbb{R}^{2d},\ \forall\, t \geq 0, \quad \pi_{z,t}(\mathbf{z}) = \exp\left(- \inf_{\varphi\in\mathbb{H}, \mathbf{z}_\varphi(z,.)=\mathbf{z}(.)} \frac{1}{2}\int_0^t |\dot{\varphi}(s)|^2 ds \right).$$

Point (i): The property that π_z is upper-compact means that for all $a \in (0,1]$, the set $K_a := \{\mathbf{z}, \pi_z(\mathbf{z}) \geq a\}$ is a compact set (for the topology of uniform convergence on compact sets). For this, we use the Ascoli Theorem. We first show the boundedness property for the paths of K_a. From the definition of π_z, we observe that for any $\mathbf{z}$ of K_a, there exists a control $\varphi \in \mathbb{H}$ such that $\mathbf{z} = \mathbf{z}_\varphi$ and such that

$$\int_0^\infty |\dot{\varphi}(s)|^2 ds \leq -2\log a + 1\,. \tag{3.2}$$

Using the above defined function $\mathcal{E}$, one checks that for all $p > 0$,

$$\begin{aligned}\frac{d}{dt}\left(\mathcal{E}^p(\mathbf{z}(t))\right) &= p\mathcal{E}(\mathbf{z}(t))^{p-1}\left(|\mathbf{y}(t)|^2 + \langle \nabla U(\mathbf{x}(t)), \dot{\varphi}(t)\rangle \right)\\ &\leq C\left(\mathcal{E}(\mathbf{z}(t))^p + \mathcal{E}(\mathbf{z}(t))^{2p-2}|\nabla U(x)|^2 + |\dot{\varphi}(t)|^2 \right).\end{aligned}$$

Under $(\mathbf{H_{Q+}})$ or $(\mathbf{H_{Q-}})$, we have respectively $|\nabla U|^2 = O(U^{2-2\rho})$ or $|\nabla U|^2 = O(U)$. Thus, applying the inequalities with $\bar{p} = \rho$ (resp. $\bar{p} = 1$) under $(\mathbf{H_{Q+}})$ (resp. $(\mathbf{H_{Q-}})$) yields:

$$\frac{d}{dt}\left(\mathcal{E}^{\bar{p}}(\mathbf{z}(t))\right) \leq C\left(\mathcal{E}^{\bar{p}}(\mathbf{z}(t)) + |\dot{\varphi}(t)|^2\right), \tag{3.3}$$

and the Gronwall Lemma implies that

$$\forall\, t > 0,\ \exists\, C_t > 0,\ \forall s \in [0,t],\ \mathcal{E}^{\bar{p}}(\mathbf{z}(s)) \leq C_t\left(\mathcal{E}^{\bar{p}}(z) + C\int_0^s |\dot{\varphi}(u)|^2 du \right). \tag{3.4}$$

Finally, Equation (3.2) combined with (3.4) and the fact that $\lim_{|z|\to+\infty} \mathcal{E}(z) = +\infty$ yields

$$\sup_{\mathbf{z}\in K_a} \sup_{s\in[0,t]} |\mathbf{z}(s)| < +\infty. \tag{3.5}$$

Now, let us prove that K_a is equicontinuous: for all $t > 0$, $u, v \in [0,t]$ with $u \le v$ and $\mathbf{z} \in K_a$, we know that for a suitable constant $\tilde{C}_{t,a,z}$, the controlled trajectories of K_a are *a priori* bounded: $\sup_{s\in[0,t]} |\mathbf{z}(s)| \le \tilde{C}_{t,a,z}$. Since b is continuous, the Cauchy-Schwarz Inequality yields:

$$|\mathbf{z}(v) - \mathbf{z}(u)| \le \int_u^v |b(\mathbf{z}(s))|ds + \int_u^v |\dot{\varphi}(s)|ds \le \sup_{|z|\le \tilde{C}_{t,a,z}} |b(z)|(v-u) + \sqrt{1-2\log a}\sqrt{v-u}.$$

The two conditions of the Ascoli Theorem being satisfied, the compactness of K_a follows.

Point (ii): We do not detail this item which easily follows from the controls established in the proof of (i) (see (3.4)). Finally, the other conditions of Theorem 5.2.12 of Puhalskii (2001) being trivially satisfied, the lemma follows. □

3.2 Exponential tightness (Proof of i) of Theorem 2.1)

In the next proposition, we investigate the exponential tightness of $(\nu_\varepsilon)_{\varepsilon\in(0,1]}$. Our approach consists in showing sufficiently sharper estimates for hitting time of the process $(Z_t^\varepsilon)_{t\ge 0}$.

Proposition 3.1 *Assume* $(\mathbf{H_{Q+}})$ *or* $(\mathbf{H_{Q-}})$*, then there exists a compact set B of $\mathbb{R}^{2d}$, such that the first hitting time τ_ε of B defined as $\tau_\varepsilon = \inf\{t > 0, Z_t^\varepsilon \in B\}$ satisfies the three properties:*

(i) For all compact set K of $\mathbb{R}^{2d}$,

$$\limsup_{\varepsilon\to 0} \sup_{z\in K} \mathbb{E}_z[(\tau_\varepsilon)^2] < \infty. \tag{3.6}$$

(ii) There exists $\delta > 0$ such that for all compact set K of $\mathbb{R}^{2d}$,

$$\limsup_{\varepsilon\to 0} \sup_{z\in K} \sup_{t\ge 0} \mathbb{E}_z \left[|Z_{t\wedge\tau_\varepsilon}^\varepsilon|^{\frac{\delta}{\varepsilon^2}} \right]^{\varepsilon^2} < +\infty. \tag{3.7}$$

(iii) For evey compact set K of $\mathbb{R}^{2d}$ such that $K \cap B = \emptyset$,

$$\liminf_{\varepsilon\to 0} \inf_{z\in K} \mathbb{E}_z[\tau_\varepsilon] > 0. \tag{3.8}$$

As a consequence, the family of invariant distributions $(\nu^\varepsilon)_{\varepsilon\in(0,1]}$ is exponentially tight.

The conclusion of the above proposition follows directly from Lemma 7 of Puhalskii (2003). A fundamental step of the proof of Proposition 3.1 is the next lemma which shows some mean-reverting properties for the process (with some constants that do not depend on ε). Its technical proof is postponed in the appendix. Note that such lemma uses a key Lyapunov function V which is rather not standard due to the kinetic form of the coupled process.

Lemma 3.2 *Assume* $(\mathbf{H_{Q+}})$ *or* $(\mathbf{H_{Q-}})$ *and let $V : \mathbb{R}^{2d} \to \mathbb{R}$ be defined by*

$$V(x,y) = U(x) + \frac{|y|^2}{2} + m\left(\frac{|x|^2}{2} - \langle x, y\rangle\right),$$

with $m \in (0,1)$. For $p > 0$, $\delta > 0$ and $\varepsilon > 0$, set

$$\psi_\varepsilon(x,y) = \exp\left(\frac{\delta V^p(x,y)}{\varepsilon^2}\right).$$

Then, if $p \in (0,1)$ under $(\mathbf{H_{Q+}})$ *and $p \in (1-a, a)$ under* $(\mathbf{H_{Q-}})$ *and δ is a positive number, there exist $\alpha, \beta, \alpha', \beta'$ positive such that for all $(x,y) \in \mathbb{R}^{2d}$ and $\varepsilon \in (0,1]$*

$$\mathcal{A}^\varepsilon V^p(x,y) \le \beta - \alpha V^{\bar{p}}(x,y) \quad \text{and,} \tag{3.9}$$

$$\mathcal{A}^\varepsilon \psi_\varepsilon(x,y) \le \frac{\delta}{\varepsilon^2}\psi_\varepsilon(x,y)(\beta' - \alpha' V^{\bar{p}}(x,y)), \tag{3.10}$$

where $\mathcal{A}^\varepsilon$ *is the infinitesimal generator of* $(X_t^\varepsilon, Y_t^\varepsilon)$ *defined in* (2.4) *and where*

$$\bar{p} = \begin{cases} p & \text{under } (\mathbf{H_{Q+}}) \\ p+a-1 & \text{under } (\mathbf{H_{Q-}}). \end{cases}$$

Proof of Proposition 3.1: For sake of simplicity, we omit the ε dependence and write (X_t, Y_t) instead of $(X_t^\varepsilon, Y_t^\varepsilon)$.

• Proof of (i): We use a Lyapunov method to bound the second moment of the hitting time τ_ε. Let $p \in (0,1)$. By the Itô formula, we have

$$\frac{V^p(X_t, Y_t)}{1+t} = V^p(x,y) + \int_0^t -\frac{V^p(x,y)}{(1+s)^2} + \frac{\mathcal{A}^\varepsilon V^p(x,y)}{1+s} ds + \varepsilon M_t, \tag{3.11}$$

where (M_t) is the local martingale defined by

$$M_t \;=\; \int_0^t p\frac{V^{p-1}(X_s,Y_s)}{1+s}\langle \nabla U(X_s) + m(X_s - Y_s), dB_s\rangle. \tag{3.12}$$

Since V is a positive function, we have that

$$\frac{1}{\varepsilon^2}\int_0^t -\frac{\mathcal{A}^\varepsilon V^p(X_s,Y_s)}{1+s}ds - \frac{1}{2}\left\langle \frac{M_t}{\varepsilon}, \frac{M_t}{\varepsilon}\right\rangle \;\leq\; \frac{1}{\varepsilon^2}V^p(x,y) + \frac{M_t}{\varepsilon} - \frac{1}{2}\left\langle \frac{M_t}{\varepsilon}, \frac{M_t}{\varepsilon}\right\rangle. \tag{3.13}$$

Note that in the above expression, the martingale $(\frac{M_t}{\varepsilon})_{t\geq 0}$ has been compensated by its stochastic bracket in order to use further exponential martingale properties. The l.h.s. of (3.13) satisfies

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\int_0^t -\frac{\mathcal{A}^\varepsilon V^p(X_s,Y_s)}{1+s}ds - \frac{1}{2}\langle \frac{M_t}{\varepsilon}, \frac{M_t}{\varepsilon}\rangle \\
&\quad = \frac{1}{\varepsilon^2}\int_0^t \frac{1}{1+s}\left(-\mathcal{A}^\varepsilon V^p(X_s,Y_s) - \frac{p^2 V^{2p-2}(X_s,Y_s)}{1+s}|\nabla U(X_s) + m(X_s - Y_s)|^2\right) ds \\
&\quad \geq \frac{1}{\varepsilon^2}\int_0^t \frac{H_{p,\varepsilon}(X_s,Y_s)}{1+s}ds.
\end{aligned}$$

with $H_{p,\varepsilon}(x,y) = -\mathcal{A}^\varepsilon V^p(x,y) - p^2 V^{2p-2}(x,y)|\nabla U(x) + m(x-y)|^2$. Then, a localization of (M_t) combined with the Fatou Lemma yields for all stopping time τ

$$\mathbb{E}\left[\exp\left(\frac{1}{\varepsilon^2}\int_0^{t\wedge\tau} \frac{H_{p,\varepsilon}(X_s,Y_s)}{1+s}ds\right)\right] \leq \exp\left(\frac{1}{\varepsilon^2}V^p(x,y)\right).$$

The final step relies on the fact that there exists $p \in (0,1)$ and $M_1 > 0$ such that:

$$\forall (x,y) \in \bar{B}(0,M_1)^c \text{ and } \forall \varepsilon \in (0,1], \quad H_{p,\varepsilon}(x,y) \geq 2. \tag{3.14}$$

Let us prove the above inequality under condition $((\mathbf{H_{Q+}}))$ or $(\mathbf{H_{Q-}})$. First, since $m \in (0,1)$, one can check that there exists $C > 0$ such that

$$\forall (x,y) \in \mathbb{R}^{2d}, \quad |x|^2 + |y|^2 \leq CV(x,y). \tag{3.15}$$

As a consequence, we have

$$\lim_{|(x,y)|\to+\infty} V(x,y) = +\infty. \tag{3.16}$$

Now, owing to the assumptions on ∇U, it follows that,

$$V^{2p-2}(x,y)|\nabla U(x) + m(x-y)|^2 = \begin{cases} O(V^{2(p-\rho)}(x,y)) + O(V^{2p-1}(x,y)) & \text{under } ((\mathbf{H_{Q+}})) \\ O(V^{2p-1}(x,y)) & \text{under } ((\mathbf{H_{Q-}})). \end{cases}$$

From now on, assume that

$$\begin{cases} 0 < p < 2\rho \wedge 1 & \text{under } ((\mathbf{H_{Q+}})) \\ 1-a < p < a & \text{under } ((\mathbf{H_{Q-}})). \end{cases} \tag{3.17}$$

By Lemma 3.2, we then obtain that for all $(x,y) \in \mathbb{R}^{2d}$ and $\varepsilon \in (0,1]$

$$H_{p,\varepsilon}(x,y) \geq -\beta + \alpha V^{\bar{p}}(x,y) - O(V^{2p-1}),$$

where $\bar{p}$ is defined in Lemma 3.2. Under (3.17), one checks that $2p-1 < \bar{p}$ and then uniformly in ε:

$$\lim_{|(x,y)| \to +\infty} H_{p,\varepsilon}(x,y) = +\infty,$$

and (3.14) follows. Next, we consider (3.6) with τ being $\tau_\varepsilon = \inf\{t \geq 0, Z_t^\varepsilon \in \bar{B}(0,M_1)\}$, where M_1 is such that (3.14) holds. We then have

$$\mathbb{E}\left[\exp\left(\frac{1}{\varepsilon^2}\int_0^{t\wedge\tau_\varepsilon} \frac{2}{1+s} ds\right)\right] \leq \mathbb{E}\left[\exp\left(\frac{1}{\varepsilon^2}\int_0^{t\wedge\tau_\varepsilon} \frac{H_{p,\varepsilon}(X_s,Y_s)}{1+s}\right) ds\right] \leq \exp\left(\frac{V^p(x,y)}{\varepsilon^2}\right).$$

Computing the integral and using the Fatou Lemma, we get

$$\mathbb{E}_{(x,y)}\left[(1+\tau_\varepsilon)^{\frac{2}{\varepsilon^2}}\right] \leq \exp\left(\frac{1}{\varepsilon^2} V^p(x,y)\right).$$

The Jensen Inequality applied to $x \to x^{\frac{1}{\varepsilon^2}}$ yields that for every $(x,y) \in \mathbb{R}^{2d}$, for all $\varepsilon \in (0,1]$

$$\mathbb{E}_{(x,y)}[(1+\tau_\varepsilon)^2] \leq \exp\left(V^p(x,y)\right).$$

The first statement follows using that V^p is locally bounded.

- Proof of (ii): Thanks to (3.15), we have for all $p > 0$ and for $|(x,y)|$ large enough,

$$\ln(|(x,y)|) \leq \frac{1}{2}\ln(CV(x,y)) \leq V^p(x,y). \tag{3.18}$$

Multiplying by δ/ε^2, this inequality suggests the computation of

$$\mathbb{E}\left[\exp\left(\frac{\delta}{\varepsilon^2} V^p(X_{t\wedge\tau}, Y_{t\wedge\tau})\right)\right],$$

for appropriate p and τ. Applying the Itô formula to the function $\psi_\varepsilon(x,y) = \exp(\delta V^p(x,y)/\varepsilon^2)$, we get for all t

$$\psi_\varepsilon(X_t,Y_t) = \psi_\varepsilon(x,y) + \int_0^t \mathcal{A}\psi_\varepsilon(X_s,Y_s)ds + M_t, \tag{3.19}$$

where $(M_t)_{t\geq 0}$ is a local martingale that we do not need to make explicit. Let us choose $p \in (0,1)$ such that inequality (3.10) of Lemma 3.2 holds. Since $V(x,y) \to +\infty$ as $|(x,y)| \to +\infty$ and since $\bar{p} > 0$, we deduce that

$$\beta' - \alpha' V^{\bar{p}}(x,y) \xrightarrow{|(x,y)|\to+\infty} -\infty.$$

As a consequence, for all positive δ, there exists $M_2 > 0$ such that

$$\forall \varepsilon \in (0,1], \quad \forall (x,y) \in \bar{B}(0,M_2)^c, \quad \mathcal{A}\psi_\varepsilon \leq 0.$$

Let $\tau_\varepsilon = \inf\{t \geq 0, (X_t,Y_t) \in \bar{B}(0,M_2)\}$, a standard localization argument in (3.19) yields

$$\mathbb{E}_{(x,y)}[\psi_\varepsilon(X_{t\wedge\tau_\varepsilon}, Y_{t\wedge\tau_\varepsilon})] \leq \psi_\varepsilon(x,y).$$

Without loss of generality, we can assume that M_2 is such that (3.18) is valid for all $(x,y) \in \bar{B}(0,M_2)^c$. It follows that for all $\varepsilon \in (0,1]$, $t \geq 0$ and $(x,y) \in \bar{B}(0,M_2)^c$,

$$\left(\mathbb{E}_{(x,y)}\left[|X_{t\wedge\tau_\varepsilon}, Y_{t\wedge\tau_\varepsilon}|^{\frac{\delta}{\varepsilon^2}}\right]\right)^{\varepsilon^2} \leq e^{\delta V^p(x,y)}.$$

From the above inequality, we finally deduce (3.7).

• Proof of (iii): With the notations of the two previous parts of the proof, the properties (3.6) and (3.7) hold with $\tau_\varepsilon := \inf\{t \geq 0, (X_t, Y_t) \in B\}$ for all compact set B such that $\bar{B}(0, M_1 \vee M_2) \subset B$. In this last part of the proof, we then set $B = \bar{B}(0, M)$ where $M \geq M_1 \vee M_2$.

Second, remark that it is enough to show that the result holds with $\tau_\varepsilon \wedge 1$ instead of τ_ε. Now, let K be a compact set of $\mathbb{R}^{2d}$ such that $B \cap K = \emptyset$ and let $(\varepsilon_n, z_n)_{n\geq 1}$ be a sequence such that $\varepsilon_n \to 0$, such that $z_n \in K$ for all $n \geq 1$ and such that

$$\mathbb{E}_{z_n}[\tau_{\varepsilon_n} \wedge 1] \xrightarrow{n\to+\infty} \liminf_{\varepsilon\to 0} \inf_{z\in K} \mathbb{E}_z[\tau_\varepsilon \wedge 1].$$

Up to an extraction, we can assume that $(z_n)_{n\geq 1}$ is a convergent sequence. Let $\tilde{z}$ denote its limit. Lemma 3.1 implies that $(\mathcal{L}((Z^{\varepsilon_n,z_n})_{t\in[0,1]})_{n\geq 1}$ is exponentially tight and then tight on $\mathcal{C}([0,1],\mathbb{R}^d)$. Using a second extraction, we can assume that $(Z^{\varepsilon_n,z_n})_{n\geq 1}$ converges in distribution to $Z^{(\infty)}$. Furthermore, since $\varepsilon_n \to 0$, the limit process $Z^{(\infty)}$ is $a.s.$ a solution of the o.d.e. $\dot{\mathbf{z}} = b(\mathbf{z})$ starting at $\tilde{z}$. The function b being locally Lipschitz continuous, uniqueness holds for the solutions of this o.d.e. and we can conclude that $(Z^{\varepsilon_n,z_n})_{n\geq 1}$ converges in distribution to $\mathbf{z}(\tilde{z},.)$ (where $\mathbf{z}(\tilde{z},.)$ denotes the unique solution of $\dot{\mathbf{z}} = b(\mathbf{z})$ starting from $\tilde{z}$). The function $\mathbf{z}(\tilde{z},.)$ being deterministic, the convergence holds in fact in probability and at the price of a last extraction, we can assume without loss of generality that $(Z^{\varepsilon_n,z_n})_{n\geq 1}$ converges $a.s.$ to $\mathbf{z}(\tilde{z},.)$. In particular, setting $\delta := d(K,B)$ $(\delta > 0)$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\sup_{t\in[0,1]} |Z_t^{\varepsilon_n,z_n} - \mathbf{z}(\tilde{z},t)| \leq \frac{\delta}{4} \quad a.s.$$

Setting now,

$$\tau_{\tilde{z},\frac{\delta}{2}} := \inf\{t \geq 0, d(\mathbf{z}(\tilde{z},t), B) \leq \frac{\delta}{2}\} \wedge 1,$$

we deduce that for all $n \geq n_0$,

$$\inf_{t\in[0,\tau_{\tilde{z},\frac{\delta}{2}}]} d(Z_t^{\varepsilon_n,z_n}, B) \geq \frac{\delta}{4} \implies \tau_{\varepsilon_n} \geq \tau_{\tilde{z},\frac{\delta}{2}} \quad a.s.$$

Using the Fatou Lemma, we can conclude that

$$\lim_{n\to+\infty} \mathbb{E}_{z_n}[\tau_{\varepsilon_n} \wedge 1] \geq \mathbb{E}_{z_n}[\liminf_{n\to+\infty} \tau_{\varepsilon_n} \wedge 1] \geq \tau_{\tilde{z},\frac{\delta}{2}}.$$

Finally, since $t \mapsto \mathbf{z}(\tilde{z},t)$ is a continuous function and since $d(K, \bar{B}(0, M + \frac{\delta}{2})) > 0$, the stopping time $\tau_{\tilde{z},\frac{\delta}{2}}$ is clearly positive. The result follows and this finishes the proof of Proposition 3.1. □

3.3 Hamilton-Jacobi equation (Proof of ii) of Theorem 2.1)

This point is a consequence of the finite time large deviation principle for $(Z^\varepsilon)_{\varepsilon\geq 0}$ (Lemma 3.1) and of the exponential tightness of $(\nu_\varepsilon)_{\varepsilon\geq 0}$ (Proposition 3.1). This is the purpose of the next proposition which is an adaptation of Corollary 1 of Puhalskii (2003).

Proposition 3.2 *For all $\varepsilon > 0$, let $(P_t^\varepsilon(z,.))_{t\geq 0, z\in\mathbb{R}^{2d}}$ denote the semi-group associated to* (2.2) *whose unique invariant distribution is denoted by ν_ε. Then, if the following assumptions hold:*

(i) $(\nu_\varepsilon)_{\varepsilon>0}$ *is exponentially tight of order* ε^{-2} *on* $\mathbb{R}^{2d}$.

(ii) For all $t \geq 0$ *and* $z \in \mathbb{R}^{2d}$*, there exists a function* $I_t(z,.) : \mathbb{R}^{2d} \to \mathbb{R}$*, such that for all* $(z_\varepsilon)_{\varepsilon>0}$ *such that* $z_\varepsilon \to z$ *as* $\varepsilon \to 0$*,* $P_t^\varepsilon(z_\varepsilon, .)$ *satisfy a LDP with speed* ε^{-2} *and rate function* $I_t(z,.)$*.*

Then, $(\nu_\varepsilon)_{\varepsilon>0}$ *admits a* (LD)*-convergent subsequence and for such subsequence* $(\nu_{\varepsilon_k})_{k\geq 0}$ *(with* $\varepsilon_k \to 0$ *as* $k \to +\infty$*), the associated rate function* W *satisfies for all* $z_0 \in \mathbb{R}^{2d}$*,*

$$\forall t \geq 0, \qquad W(z_0) = \inf_{z\in\mathbb{R}^{2d}} \left(I_t(z,z_0) + W(z)\right). \tag{3.20}$$

With the terminology of Puhalskii (2003), Equation (3.20) means that $\tilde{W}$ defined for all $\Gamma \in \mathcal{B}(\mathbb{R}^{2d})$ by $\tilde{W}(\Gamma) = \sup_{y\in\Gamma}\exp(-W(y))$ is an invariant deviability for $(P_t^\varepsilon(z,.))_{t\geq 0, z\in\mathbb{R}^{2d}}$. In Corollary 1 of Puhalskii (2003), this result is stated with a uniqueness assumption on the invariant deviabilities. The above proposition is in fact an extension of this corollary to the case where uniqueness is not fulfilled. We refer to Appendix A for details.

Owing to Proposition 3.1 and Lemma 3.1, Proposition 3.2 can be applied with $I_t(z,.)$ defined in (3.1). The rate function W is solution of (3.20) and Equation (2.7) is satisfied. Thus, the next result proves the assertion *ii)* of Theorem 2.1.

Proposition 3.3 *Assume that either* $(\mathbf{H_{Q+}})$ *or* $(\mathbf{H_{Q-}})$ *is satisfied, then any (good) rate function* W *associated to any (LD)-convergent subsequence* $(\varepsilon_n)_{n\geq 1}$ *satisfies for all* $t \geq 0$ *and for all* $z \in \mathbb{R}^{2d}$

$$W(z_0) = \inf_{\begin{cases}\varphi \in \mathbb{H}\\ \tilde{\mathbf{z}}_\varphi(0) = z_0\end{cases}} \left[\frac{1}{2}\int_0^t |\dot{\varphi}(s)|^2 ds + W(\tilde{\mathbf{z}}_\varphi(t))\right].$$

<u>*Proof :*</u> We know that W satisfies (3.20) and thus for any $z_0 \in \mathbb{R}^{2d}$

$$W(z_0) = \inf_{v\in\mathbb{R}^{2d}} \left(I_t(v,z_0) + W(v)\right) = \inf_{\begin{cases}v \in \mathbb{R}^{2d}, \varphi \in \mathbb{H}\\ \mathbf{z}_\varphi(0) = v, \mathbf{z}_\varphi(t) = z_0\end{cases}} \left[\frac{1}{2}\int_0^t |\dot{\varphi}(s)|^2 ds + W(\mathbf{z}_\varphi(0))\right].$$

Remarking that $g : [0,t] \to \mathbb{R}^{2d}$ defined by $g(s) = \mathbf{z}_\varphi(t-s)$ is a controlled trajectory associated to $-b$ and $-\varphi$, we deduce that for all $t \geq 0$

$$W(z_0) = \inf_{\begin{cases}v \in \mathbb{R}^{2d}, \varphi \in \mathbb{H}\\ \tilde{\mathbf{z}}_{-\varphi}(0) = z_0, \tilde{\mathbf{z}}_{-\varphi}(t) = v\end{cases}} \left[\frac{1}{2}\int_0^t |\dot{\varphi}(s)|^2 ds + W(\tilde{\mathbf{z}}_{-\varphi}(t))\right].$$

The result follows from the change of variable $\tilde{\varphi} = -\varphi$. $\square$

3.4 Infinite horizon Hamilton-Jacobi equation

The aim of this part is to show that when there is a finite number of critical points, we can "replace t by $+\infty$" in (2.7). This proof is an adaptation of Theorem 4 of Biswas and Borkar (2009). The main novelty of our proof is the second step. Indeed, using arguments based on asymptotic pseudo-trajectories and Lyapunov functions, we prove that the optimal controlled trajectory is attracted by a critical point of the drift vector field.

Proof of (iii) **of Theorem 2.1:** The proof is divided in three parts. We first build an optimal path $t \mapsto \tilde{z}_\psi(z,t)$ for the Hamilton Jacobi equation of interest. Then, we focus in the second step on its long time behaviour and obtain that $\tilde{z}_\psi(z,t)$ converges to $z^\star$ which belongs to $\{z, b(z) = 0\}$.

In order to conclude, we need to prove the continuity of W at each point of $\{z, b(z) = 0\}$. This is the purpose of the third step.

• Step 1: We show that we can build a function $\psi \in \mathbb{H}$ such that for all $z \in \mathbb{R}^{2d}$ the couple $(\tilde{\mathbf{z}}_\psi(z,t), \dot{\psi}(t))_{t\geq 0}$ on $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^{2d}) \times L^{2,loc}(\mathbb{R}_+, \mathbb{R}^d)$ satisfies for all $t > 0$,

$$W(z) = \frac{1}{2}\int_0^t |\dot{\psi}(s)|^2 ds + W(\tilde{\mathbf{z}}_\psi(z,t)). \tag{3.21}$$

First, let $T > 0$ and let $(\tilde{\mathbf{z}}^{(n)}_{\varphi^{(n)}}, \varphi^{(n)})_{n\geq 1}$ be a minimizing sequence of $\mathcal{C}([0,T], \mathbb{R}^{2d}) \times \mathbb{H}$ such that

$$\frac{1}{2}\int_0^T |\dot{\varphi}^{(n)}(s)|^2 ds + W(\tilde{\mathbf{z}}^{(n)}_{\varphi^{(n)}}(z,T)) \xrightarrow{n\to+\infty} \inf_{\varphi\in\mathbb{H}, \mathbf{z}_\varphi(0)=z} \frac{1}{2}\int_0^T |\dot{\varphi}(s)|^2 ds + W(\tilde{\mathbf{z}}_\varphi(z,T)).$$

Since W is non negative, it is clear that $(\int_0^T |\dot{\varphi}^{(n)}(s)|^2 ds)_{n\geq 0}$ is bounded. It follows that $(\dot{\varphi}^{(n)})_{n\geq 1}$ is relatively compact on $L^2_w([0,T], \mathbb{R}^{2d})$ which denotes the set of square-integrable functions on $[0,T]$ endowed with the weak topology. This also implies that

$$M := \sup_{n\geq 1} \sup_{t\in[0,T]} |\tilde{\mathbf{z}}^{(n)}_{\varphi^{(n)}}(t)| < +\infty. \tag{3.22}$$

Actually, under $(\mathbf{H_{Q-}})$, b is Lipschitz continuous and this point is classical. Now, under $(\mathbf{H_{Q+}})$, we have $|\nabla U| = O(U^{1-\rho})$ with $\rho \in (0,1)$. Since $\sup_{n\geq 1} \int_0^T |\dot{\varphi}^{(n)}(s)|^2 ds < +\infty$, Inequality (3.4) implies that

$$\sup_{n\geq 1} \sup_{t\in[0,T]} \mathcal{E}^\rho \left(\tilde{\mathbf{z}}^{(n)}_{\varphi^{(n)}}(t)\right) < +\infty.$$

Since $\lim_{|z|\to+\infty} \mathcal{E}(z) = +\infty$, Equation (3.22) also follows in this case.

Now, since b is locally Lipschitz, b is then Lipschitz continuous on $B(0,M)$ and a classical argument based on the Ascoli Theorem shows that $(\tilde{\mathbf{z}}^{(n)}_{\varphi^{(n)}})_{n\geq 1}$ is relatively compact on $\mathcal{C}([0,T], \mathbb{R}^{2d})$. It follows that $(\tilde{\mathbf{z}}^{(n)}_{\varphi^{(n)}}, \dot{\varphi}^{(n)})_{n\geq 1}$ is relatively compact on $\mathcal{C}([0,T], \mathbb{R}^{2d}) \times L^2_w([0,T], \mathbb{R}^{2d})$ and thus there exists a convergent subsequence to $(\hat{\mathbf{z}}^T, \dot{\psi}_T)$ which belongs to $\mathcal{C}([0,T], \mathbb{R}^{2d}) \times L^2_w([0,T], \mathbb{R}^{2d})$. Using that b is a continuous function, one checks that $\hat{\mathbf{z}}^T(t) = \tilde{\mathbf{z}}_{\psi_T}(z,t)$, for all $t \in [0,T]$ and the couple $(\hat{z}^T, \dot{\psi}_T)$ satisfies (3.21) (for a fixed T). Furthermore, for all $t \in [0,T]$, we have

$$W(\tilde{\mathbf{z}}_{\psi_T}(z,t)) = \frac{1}{2}\int_t^T |\dot{\psi}_T(s)|^2 ds + W(\tilde{\mathbf{z}}_{\psi_T}(z,T)), \tag{3.23}$$

and then (3.21) holds for all $t \in [0,T]$. As a consequence, we can build $(\tilde{\mathbf{z}}_\psi(z,.), \dot{\psi}) \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}^{2d}) \times L^{2,loc}(\mathbb{R}_+, \mathbb{R}^{2d})$ (by concatenation) which satisfies (3.21) (for all $t \geq 0$).

• Step 2: Dropping the initial condition z, we show that $(\tilde{\mathbf{z}}_\psi(t+.))_{t\geq 0}$ converges as $t \to +\infty$ to a stationary solution of $\dot{\mathbf{z}} = -b(\mathbf{z})$. First, as in (3.23),

$$W(\tilde{\mathbf{z}}_\psi(t+s)) - W(\tilde{\mathbf{z}}_\psi(t)) = -\frac{1}{2}\int_t^{t+s} |\dot{\psi}(u)|^2 du \tag{3.24}$$

and it follows that $(W(\tilde{\mathbf{z}}_\psi(t))_{t\geq 0}$ is a non-increasing and thus bounded function. Since W is a good rate function, for all $M > 0$, $W^{-1}([0,M])$ is a compact subset of $\mathbb{R}^{2d}$. This means that $(\tilde{\mathbf{z}}_\psi(t))_{t\geq 0}$ is bounded. From (3.24), we deduce that $(\int_t^{t+T} |\dot{\psi}(s)|^2 ds)_{t\geq 0}$ is also bounded. Thus, as in Step 1, owing to the previous statements and to the fact that b is locally Lipschitz continuous, we deduce from the Ascoli Theorem that $(\tilde{\mathbf{z}}_\psi(t+.))$ is relatively compact (for the topology of uniform convergence on compact sets).

We denote now by $\tilde{\mathbf{z}}^\infty_\psi(.)$ the limit of a convergent subsequence. Let us show that $(\tilde{\mathbf{z}}^\infty_\psi(t))_{t\geq 0}$ is a solution of $\dot{\mathbf{z}} = -b(\mathbf{z})$. First, since $(W(\tilde{\mathbf{z}}_\psi(t)))_{t\geq 0}$ is non-increasing (and non negative as a rate function), we again deduce from (3.24) that $\int_t^{t+T} |\dot{\psi}(u)|^2 du \xrightarrow{t\to+\infty} 0$. As a consequence, using that for all $s \geq 0$, the map $\mathbf{z} \mapsto \mathbf{z}(s) - \mathbf{z}(0) + \int_0^s b(\mathbf{z}(u))du$ (from $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^{2d})$ to $\mathbb{R}^{2d}$) is continuous and that

$$\forall\, t,\, T \geq 0 \text{ and } s \in [0,T], \left|\tilde{\mathbf{z}}_\psi(t+s) - \tilde{\mathbf{z}}_\psi(t) + \int_0^s b(\tilde{\mathbf{z}}_\psi(t+u))du\right| \leq C_T \left(\int_t^{t+T} |\dot{\psi}(u)|^2 du\right)^{\frac{1}{2}},$$

we obtain that $(\tilde{\mathbf{z}}_\psi^\infty(t))_{t\geq 0}$ is a solution of $\dot{\mathbf{z}} = -b(\mathbf{z})$. It remains to show that $(\tilde{\mathbf{z}}_\psi^\infty(t))_{t\geq 0}$ is stationary, *i.e.* that every limit point of $(\tilde{\mathbf{z}}_\psi(t))_{t\geq 0}$ belongs to $\{z \in \mathbb{R}^{2d}, b(z) = 0\}$. Denote by $(\Phi_t(z))_{t,z}$ the flow associated to the o.d.e. $\dot{\mathbf{z}} = -b(\mathbf{z})$. Again, owing to the fact that for all $T > 0$, $\int_t^{t+T} |\dot{\psi}(u)|^2 du \xrightarrow{t\to+\infty} 0$, we can deduce that for all $T > 0$,

$$\sup_{s\in[0,T]} |\tilde{\mathbf{z}}_\psi(t+s) - \Phi_s(\tilde{\mathbf{z}}_\psi(t))| \xrightarrow{t\to+\infty} 0.$$

This means that $(\tilde{\mathbf{z}}_\psi(t))_{t\geq 0}$ is an *asymptotic pseudo-trajectory* for Φ (see (Benaim, 1996)). As a consequence, by Proposition 5.3 and Theorem 5.7 of Benaim (1996), the set K of limit points of $(\tilde{\mathbf{z}}_\psi(t))_{t\geq 0}$ is a (compact) invariant set for Φ such that $\Phi|_K$ has no proper attractor. This means that there is no strict invariant subset A of K such that for all $z \in K$, $d(\Phi_t(z), A) \xrightarrow{t\to+\infty} 0$.
It follows that in order to conclude that K is included in $\{z, b(z) = 0\}$, it is now enough to show that $A = \{z, b(z) = 0\} \cap K$ is an attractor for $\Phi|_{\bar{K}}$. To this end for a positive ρ, we consider $L : \mathbb{R}^{2d} \mapsto \mathbb{R}$ defined by

$$L(z) = U(x) + (1-\rho)\frac{|y|^2}{2} - \rho\langle\nabla U(x), y\rangle \quad \text{with } z = (x,y).$$

If $\mathbf{z}$ is solution of $\dot{\mathbf{z}} = -b(\mathbf{z})$, we have :

$$\frac{d}{dt}L(\mathbf{z}(t)) = \mathbf{y}(t)^t\Big((1-\rho)I_d - \rho D^2U(\mathbf{x}(t))\Big)\mathbf{y}(t) + \rho|\nabla U(\mathbf{x}(t))|^2.$$

Since K is a bounded invariant set and that D^2U is locally bounded, we can choose ρ small enough and $\alpha_\rho > 0$ such that for all $(\mathbf{z}(t))$ solution of $\dot{\mathbf{z}} = -b(\mathbf{z})$ with $\mathbf{z}(0) \in K$,

$$\frac{d}{dt}L(\mathbf{z}(t)) \geq \alpha_\rho|\mathbf{y}(t)|^2 + \rho|\nabla U(\mathbf{x}(t))|^2. \tag{3.25}$$

For all starting point $z \in K$, the function $t \mapsto L(\mathbf{z}(t))$ is then non-decreasing and thus convergent to $\ell_\infty \in \mathbb{R}$. Since $(\mathbf{z}(t))_{t\geq 0}$ is bounded, an argument similar to the one developed in Step 1 combined with the Ascoli Theorem yields that $(\mathbf{z}(t+.))$ is relatively compact. If $(\mathbf{z}(t_n+.))_{n\geq 0}$ denotes a subsequence of $(\mathbf{z}(t+.))$, we can assume (at the price of a potential extraction) that $(\mathbf{z}(t_n+.))_{n\geq 0}$ converges to $\mathbf{z}^\infty(.)$. We have necessarily $L(\mathbf{z}^\infty(t)) = \ell_\infty$ for all $t \geq 0$ and thus

$$\frac{d}{dt}L(\mathbf{z}^\infty(t)) = 0.$$

By (3.25), we deduce that $\mathbf{y}^\infty(t) = \nabla U(\mathbf{x}^\infty(t)) = 0$. This means that $\mathbf{z}^\infty(.)$ is a stationary solution and that every limit point of $(\mathbf{z}(t))_{t\geq 0}$ is an equilibrium point of the o.d.e.

Thus, we can conclude that every limit point of $(\tilde{\mathbf{z}}_\psi(t))_{t\geq 0}$ belongs to $\{z, b(z) = 0\}$. Finally, since the set of limit points of $(\tilde{\mathbf{z}}_\psi(t))_{t\geq 0}$ is compact connected and $\{x, \nabla U(x) = 0\}$ is finite, it follows that $\tilde{\mathbf{z}}_\psi(t) \to z^\star = (x^\star, 0)$ as $t \to +\infty$ where $x^\star \in \{x, \nabla U(x) = 0\}$. Then, by (3.21), we will deduce the announced result if we prove that W is continuous at $z^\star$. This is the purpose of the next step.

• Step 3: We prove that for all $z^\star \in \{z, b(z) = 0\}$, *i.e.* for all $z^\star = (x^\star, 0)$ with $x^\star \in \{x, \nabla U(x) = 0\}$, W is continuous at $z^\star$. Since $D^2U(x^\star)$ is invertible, we deduce from Lemma 4.1 (available in the appendix) that the dynamical system is *locally controllable* around $z^\star$, *i.e.* that for all $T > 0$, for all $\varepsilon > 0$, there exists $\eta > 0$ such that for all $z \in B(z^\star, \eta)$, $I_T(z, z^\star) \leq \varepsilon$ and $I_T(z^\star, z) \leq \varepsilon$. Now, the definition of W implies that $W(z^\star) \leq W(z) + \varepsilon$ and $W(z) \leq W(z^\star) + \varepsilon$. The continuity of W follows and it ends the proof of *(iii)* in Theorem 2.1 by letting t go to $+\infty$ in (3.21). □

4 Freidlin and Wentzell theory

In this section, we establish some sharp estimations of the behaviour of (ν_ε) (as $\varepsilon \to 0$). To this end, we follow the roadmap of Freidlin and Wentzell (1979). Our goal is twofold: first, we aim to obtain some uniqueness property for the rate function W defined in Theorem 2.1 and thus to derive a large deviation principle for (ν_ε). Second, we want to obtain a more explicit formulation of W in order to characterize at least in some particular cases, the limit behaviour of (ν_ε) for some non-convex potential U. In the rest of the paper, we will assume that the potential U satisfies Assumption $(\mathbf{H_D})$ defined in Section 2.4.1, the set of critical points of U is thus finite and we set $\{x \in \mathbb{R}^d, \nabla U(x) = 0\} = \{x_1^\star, \ldots, x_\ell^\star\}$.

First, we classify the critical points, that is, we link the critical points of the vector field b with those of U and we determine their nature (stable or unstable). Next, with respect to these critical points, we construct the so-called skeleton Markov chain associated to the process $(X_t^\varepsilon, Y_t^\varepsilon)$. With all these ingredients, we finally derive the LDP for (ν_ε).

4.1 Classification of critical points.

We first need to classify the equilibria of the dynamical system $\dot{\mathbf{z}} = b(z)$. We recall that $\{z \in \mathbb{R}^{2d}, b(z) = 0\} = \{z_1^\star, \ldots, z_\ell^\star\}$ where for all $i \in \{1, \ldots, \ell\}$, $z_i^\star = (x_i^\star, 0)$. The following proposition characterizes the nature of $z_i^\star$ with respect to that of $x_i^\star$.

Proposition 4.1 *Assume that $D^2U(x_i^\star)$ is invertible for all $i \in \{1, \ldots, \ell\}$. Then, if $x_i^\star$ is a minimum of U, then $z_i^\star$ is a stable equilibrium of the deterministic dynamical system. Otherwise, $z_i^\star$ is an unstable equilibrium.*

Proof : We denote by I the minima of U and by J the other critical points. Let us compute the Jacobian matrix of the vector field b: for each $i \in \{1, \ldots \ell\}$

$$Db(z_i^\star) = \begin{pmatrix} 0 & -I_d \\ D^2U(x_i^\star) & -I_d \end{pmatrix}.$$

Now, simple linear algebra yields the characterization of the spectrum of the linearized vector field near each equilibrium $z_i^\star$:

$$\begin{aligned} Sp(Db(z_i^\star)) &= \{\lambda, -\lambda(\lambda+1) \in Sp(D^2U(x_i^\star))\} \\ &= \left\{-1/2 \pm \sqrt{1/4 - \mu},\ \mu \in Sp(D^2U(x_i^\star))\right\}, \end{aligned}$$

where $\sqrt{1/4-\mu}$ denotes $i\sqrt{|1/4-\mu|}$ if $1/4 - \mu \leq 0$. Since $D^2U(x_i^\star)$ is a positive definite matrix, it follows that when $x_i^\star$ is a local minima of U, $\mu \in Sp(D^2U(x_i^\star))$ is then positive and

$$\forall i \in I \qquad \Re\left(Sp(Db(z_i^\star))\right) \subset (-1, 0).$$

Hence, $z_i^\star$ is a stable equilibrium when $x_i^\star$ is a local minimum of U.
When $x_i^\star$ is another equilibrium point, $D^2U(x_i^\star)$ has some negative eigenvalues μ. Then, $Db(z_i^\star)$ has some positive eigenvalues (since $\sqrt{1/4-\mu} < 1/2$ in this case) and $z_i^\star$ is thus an unstable equilibrium of the deterministic dynamical system. This ends the proof of the proposition. □

4.2 Skeleton representation

The Freidlin and Wentzell (1979) description of the invariant measure ν_ε of the continuous time Markov process strongly depends on its representation using the invariant measure of a specific skeleton Markov chain. This formula, due to Khas'minskiĭ (see Khas'minskiĭ (1980), chapter 4) in the uniform elliptic case, will remain true in our framework even if the original process is hypoelliptic and defined on a non compact manifold. This is the purpose of Proposition 4.2 below but before a precise statement, we first need to define the skeleton Markov chain associated to our process.

Let ρ_0 be the half of the minimum distance between two critical points:

$$\rho_0 = \frac{1}{2} \min_{i \neq j} d(z_i^\star, z_j^\star). \tag{4.1}$$

Now, let $0 < \rho_1 < \rho_0$ and set $g_i = B(z_i^\star, \rho_1)$. Each boundary ∂g_i is smooth as well as the one of the set g defined as

$$g = \cup_i g_i. \tag{4.2}$$

Note that by construction, $g_i \cap g_j = \emptyset$ if $i \neq j$. Finally, we denote by Γ the complementary set of the ρ_0-neighbourhood of the set of the critical points $z_i^\star$:

$$\Gamma = \left(\mathbb{R}^d \times \mathbb{R}^d\right) \setminus \cup_i B(z_i^\star, \rho_0). \tag{4.3}$$

We provide in Figure 1 a short summary of the construction of the sets $(g_i)_i, g, \Gamma$ as well as the positions of the critical points $z_i^\star$. It also provides an example of a trajectory $(Z_t^{\varepsilon,z})_{t\geq 0}$ (K will be defined in the sequel).

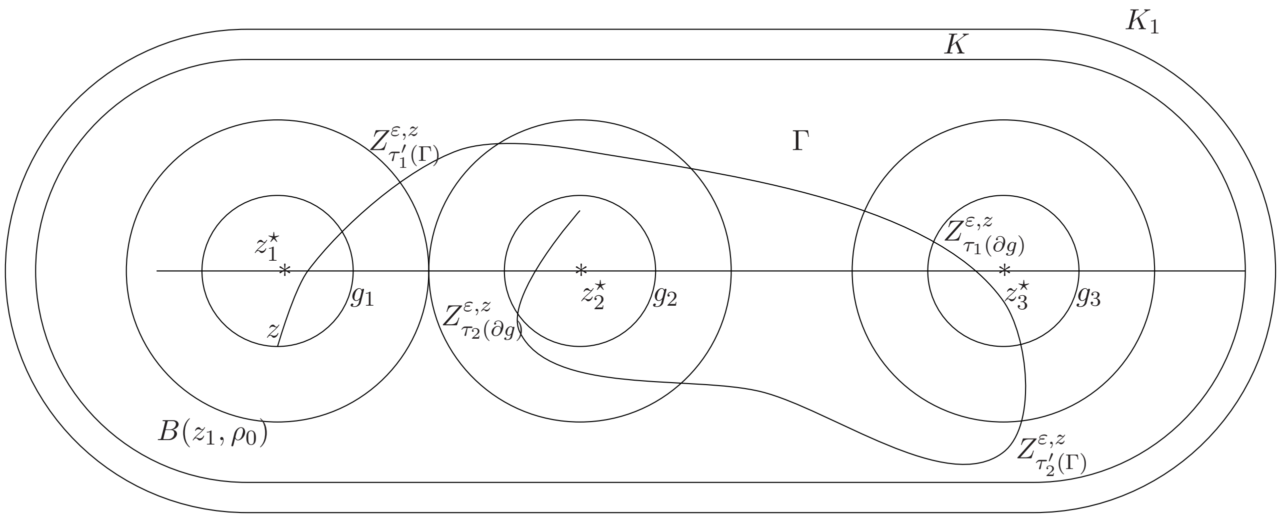

Figure 1: Graphical representation of the neighbourhood g_i, the process $(Z_t^{\varepsilon,z})_{t\geq 0}$, the skeleton chain and the compact sets K and K_1.

Now, we consider any initialisation on the boundary of the neighbourhoods of critical points $z \in \partial g$ (in our figure, $Z_0^{\varepsilon,z} = z \in \partial g_1$), and we define $(\tilde{Z}_n)_{n\in\mathbb{N}}$ the skeleton Markov chain which lives in ∂g through the classical construction of hitting and exit times of the neighbourhoods defined above. First, we set $\tau_0(\partial g) = 0$ and we also define

$$\tau'_1(\Gamma) = \inf\{t \geq 0,\ Z_t^{\varepsilon,z} \in \Gamma\}, \quad \tau_1(\partial g) = \inf\{t > \tau'_1(\Gamma),\ Z_t^{\varepsilon,z} \in \partial g\}. \tag{4.4}$$

We then follow the natural recursion

$$\tau'_n(\Gamma) = \inf\{t > \tau_{n-1}(\partial g),\ Z_t^{\varepsilon,z} \in \Gamma\}, \quad \tau_n(\partial g) = \inf\{t > \tau'_n(\Gamma),\ Z_t^{\varepsilon,z} \in \partial g\}.$$

We will show in Proposition 4.2 that for all $n \geq 0$, $\tau_n(\partial g) < +\infty$ $a.s.$ The skeleton is then defined for all $n \in \mathbb{N}$ by, $\tilde{Z}_n = Z^{\varepsilon,z}_{\tau_n(\partial g)}$. Note that $(\tilde{Z}_n)_{n\geq 0}$ belongs to ∂g and that $(\tilde{Z}_n)_{n\geq 0}$ is a Markov chain (this is actually a consequence of the strong Markov property). The set ∂g being compact, existence holds for the invariant distribution $(\tilde{Z}_n)_{n\in\mathbb{N}}$. We denote such distribution by $\tilde{\mu}_\varepsilon^{\partial g}$. The proposition states that ν_ε may be related to $\tilde{\mu}_\varepsilon^{\partial g}$.

Proposition 4.2 *(i) Following the notations introduced before, we have*

$$\forall \varepsilon > 0 \qquad \sup_{z\in\partial g} \mathbb{E}_z^\varepsilon[\tau_1(\partial g)] < \infty.$$

(ii) For any borelian set $A \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d)$ and for any $\rho_1 \in (0, \rho_0)$ the measure

$$\mu_\varepsilon^{\partial g}(A) = \int_{\partial g} \tilde{\mu}_\varepsilon^{\partial g}(dz)\mathbb{E}_z \int_0^{\tau_1(\partial g)} \mathbf{1}_{Z_s^{\varepsilon,z}\in A} ds \tag{4.5}$$

is invariant for the process $(Z_t^\varepsilon)_{t\geq 0}$. Hence, $\mu_\varepsilon^{\partial g}$ is a finite measure proportional to ν_ε.

Proof : We first prove (i). Owing to Proposition 3.1, we first check that one can find a compact set K such that $g \subseteq K$ an such that for all compact set K_1 such that $K \subseteq K_1$, the first hitting time $\tau(K)$ of K satisfies

$$\sup_{z\in\partial K_1} \mathbb{E}_z^\varepsilon[\tau(K)] < +\infty. \tag{4.6}$$

Then, the idea of the proof is to extend to our hypoelliptic context the proofs of Lemma 4.1 and 4.3 of Khas'minskiĭ (1980) given under some elliptic assumptions. Let $z \in \partial g$ and set $\tilde{\tau}_0 = \inf\{t \geq 0,\ Z_t^{\varepsilon,z} \in \partial K\}$,

$$\tilde{\tau}_1' = \inf\{t > \tilde{\tau}_0,\ Z_t^{\varepsilon,z} \in \partial g \cup \partial K_1\}, \quad \tilde{\tau}_1 = \inf\{t > \tilde{\tau}_1',\ Z_t^{\varepsilon,z} \in \partial K\},$$

and recursively for all $n \geq 2$,

$$\tilde{\tau}_n' = \inf\{t > \tilde{\tau}_{n-1},\ Z_t^{\varepsilon,z} \in \partial g \cup \partial K_1\} \quad \tilde{\tau}_n = \inf\{t > \tilde{\tau}_n',\ Z_t^{\varepsilon,z} \in \partial K\}.$$

By construction, we have $a.s.$:

$$\tau_1(\partial g) \leq \inf\{\tilde{\tau}_k',\ Z_{\tilde{\tau}_k'}^{\varepsilon,z} \in \partial g\}.$$

Then, by the strong Markov property and (4.6), it follows from a careful adaptation of the proofs of Lemma 4.1 and 4.3 of Khas'minskiĭ (1980) that $\sup_{z\in\partial g} \mathbb{E}_z^\varepsilon[\tau_1(\partial g)] < \infty$ if the two following points hold for all $\varepsilon > 0$:

- $\sup_{z\in K} \mathbb{E}_z^\varepsilon[\tau(\partial K_1)] < +\infty$.
- $\sup_{z\in K\setminus g} p_\varepsilon(z) < 1$ where $p_\varepsilon(z) := \mathbb{P}(Z_{\tau(\partial g\cup\partial K_1)}^{\varepsilon,z} \in \partial K_1)$.

As concerns the first point, it follows from Remark 5.2 of Stroock and Varadhan (1972) that it is enough to check that there exists $T > 0$, a control $(\varphi(t))_{t\in[0,T]}$ such that

$$\forall z \in K, \quad \inf\{t \geq 0, \mathbf{z}_\varphi(t,z) \in K_1^c\} \leq T. \tag{4.7}$$

Indeed, in this case, using the support theorem of Stroock and Varadhan (1970), we obtain that $\sup_{z\in K} \mathbb{P}(\tau(\partial K_1) \leq T) < 1$ and the first point follows from the strong Markov property (see Remark 5.2 of Stroock and Varadhan (1972) for details). Now, we build $(\varphi(t))_{t\geq 0}$ as follows. Let us consider the system:

$$\begin{cases} \dot{\mathbf{x}} = I_d \\ \dot{\mathbf{y}} = \nabla U(\mathbf{x}) - \mathbf{y} \end{cases}.$$

Setting $\dot{\varphi} = \mathbf{y} + I_d$, we obtain a controlled trajectory $\mathbf{z}_\varphi(z,.)$ and it is clear from its design that for all $M > 0$, there exists $T > 0$, such that for all $z \in K$, $|\mathbf{x}_\varphi(T)| > M$. The first point easily

follows.
It is well-known (see for instance Stroock and Varadhan (1972)) that for all $\varepsilon > 0$, p_ε is a solution of

$$\mathcal{A}^\varepsilon p_\varepsilon = 0 \text{ with } p_\varepsilon|_{\partial g} = 0 \text{ and } p_\varepsilon|_{\partial K_1} = 1. \tag{4.8}$$

Thus, since $\sup_{z\in K_1\backslash g} \mathbb{E}[\tau(\delta g \cup \delta K_1)] < +\infty$, since h defined by $h(x) = 1$ on ∂K_1 and $h(x) = 0$ on ∂g is obviously continuous on $\partial g \cup \partial K_1$, we can apply Theorem 9.1 of Stroock and Varadhan (1972) with $k = f = 0$ to obtain that $z \mapsto p_\varepsilon(z)$ is a continuous map. Furthermore, for all $z \in K\backslash g$, we can build a controlled trajectory starting at any $z \in \partial K$ which hit ∂g before ∂K_1. Taking for instance $\dot\varphi = 0$, we check that $(\mathcal{E}(\mathbf{x}_0(t), \mathbf{y}_0(t)))_{t\geq 0}$ is non-increasing (with $\mathcal{E}(x, y) = U(x) + |y|^2/2$) and that the accumulation points of $(\mathbf{x}_0(t), \mathbf{y}_0(t))$ lie in $\{z, b(z) = 0\}$. Thus, taking K_1 large enough in order that $\sup_{(x,y)\in K} \mathcal{E}(x, y) < \inf_{(x,y)\in K_1^c} \mathcal{E}(x, y)$, leads to an available control for all $z \in K$. Finally, using again the support theorem of Stroock and Varadhan (1970) implies that for each $z \in \partial K$, $p_\varepsilon(z) < 1$. The second point then follows from the continuity of $z \mapsto p_\varepsilon(z)$. This ends the proof of i).

Regarding now the second point (ii), as argued in the paragraph before the statement of this proposition, $(\tilde Z_n)_{n\in\mathbb{N}}$ possesses a unique invariant measure $\tilde\mu_\varepsilon^{\partial g}$. The fact that $\mu_\varepsilon^{\partial g}$ is invariant for $(Z_t^\varepsilon)_{t\geq 0}$ is standard and relies on the strong Markov property of the process (see *e.g.* in Has'minskii (1960)). □

Remark 4.1 *One could also have used an uniqueness argument for viscosity solutions to obtain the continuity of $z \mapsto p_\varepsilon(z)$ using the maximum principle on $\mathcal{A}^\varepsilon$ (as it is already used by Stroock and Varadhan (1972)). One may refer to Barles (1994) for further details.*

4.3 Transitions of the skeleton Markov chain

This paragraph is devoted to the description of estimations obtained through the Freidlin and Wentzell theory for the Markov skeleton chain defined above. These estimations as well as Proposition 4.2 are then used to obtain the asymptotic behaviour of ν_ε. In view of Theorem 2.1, we know that there exists a subsequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that (ν_{ε_n}) satisfies a large deviation principle of rate ε_n^2 with good rate function W. We then consider in the sequel this extracted subsequence but keep the notation ε. Hence, $\varepsilon \to 0$ means $\varepsilon_n \to 0$ as $n \to +\infty$ with our appropriate subsequence along with the large deviation principle holds. In the same way, ε small enough will correspond to n large enough.

4.3.1 Controllability and exit times estimates

In order to obtain some estimates related to the transition of skeleton Markov chain, the first step is to control the exit times of some balls $B(z_i^\star, \delta)$ where $z_i^\star$ denotes a critical point of $\dot{\mathbf{z}} = b(\mathbf{z})$ (similarly to Section 1, Chapter 6 of Freidlin and Wentzell (1979)). In our hypoelliptic framework, such controls of the exit times are strongly based on the controllability around the equilibria. We have the following property:

Lemma 4.1 *Let $i \in \{1, \ldots, \ell\}$ such that $D^2U(x_i^\star)$ is invertible. Let $T > 0$. Then, for all $\delta > 0$, there exists $\rho(\delta) > 0$ small enough such that*

$$\forall\, (a, b) \in B(z_i^\star, \rho(\delta)),\ \exists \varphi \in \mathbb{H} \text{ such that } \boldsymbol{z}_\varphi(a, T) = b \text{ and } \int_0^T |\dot\varphi(s)|^2 ds \leq \delta.$$

<u>*Proof :*</u> Setting

$$A = \begin{pmatrix} 0 & -I_d \\ D^2U(x_i^\star) & -I_d \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix},$$

the linearized system (at $z_i^\star$) associated with the controlled system

$$\dot{\mathbf{z}} = b(\mathbf{z}) + \begin{pmatrix} \dot{\varphi} \\ 0 \end{pmatrix} \tag{4.9}$$

can be written $\dot{\mathbf{z}} = A\mathbf{z} + Bu$ where $u = (\dot{\varphi}, \psi)^t$ with $\psi \in \mathbb{H}(\mathbb{R}^d)$. Using that $D^2U(x_i^\star)$ is invertible, one easily checks that $\mathrm{Span}(Bu, ABu, u \in \mathbb{R}^{2d}) = \mathbb{R}^{2d}$. As a consequence, the Kalman condition (see e.g. (Coron, 2007)) is satisfied and it follows from Theorems 1.16 and 3.8 of (Coron, 2007) that the system (4.9) is *locally exactly controllable* at $z_i^\star$. The lemma is then proved. $\square$

We are now able to obtain the following estimation.

Lemma 4.2 *Assume that* $(\mathbf{H_D})$ *holds and that either* $(\mathbf{H_{Q+}})$ *or* $(\mathbf{H_{Q-}})$ *is satisfied. Then, for all* $\gamma > 0$*, there exists* $\delta > 0$ *and* ε_0 *small enough such that if we define* $G = B(z_i^\star, \delta)$*, the first exit time of* G *denoted* τ_{G^c} *satisfies*

$$\forall \varepsilon \in (0, \varepsilon_0], \quad \sup_{z \in G} \mathbb{E}_z^\varepsilon \tau_{G^c} < e^{\gamma\varepsilon^{-2}}.$$

Proof : Let $i \in \{1 \ldots \ell\}$ and fix any $\gamma > 0$. By Lemma 4.1 applied with $T = 1$, one can find $\rho > 0$ such that

$$\forall (a, b) \in B(z_i^\star, 2\rho),\ \exists \varphi \in \mathbb{H} \text{ such that } \mathbf{z}_\varphi(a, 1) = b \text{ and } \frac{1}{2}\int_0^1 |\dot{\varphi}(s)|^2 ds \le \frac{\gamma}{2}.$$

Now, we set $\delta = \rho/2$, $G = B(z_i^\star, \delta)$ and we fix $a = z$ and take b such that $|z_i^\star - b| = \rho$ so that for all $z \in B(z_i^\star, \delta)$, $|z - b| \le 3\delta < \rho$. Thus, for all $z \in B(z_i^\star, \delta)$, we can find $\varphi_z \in \mathbb{H}$ such that $\mathbf{z}_{\varphi_z}(z, 1) = b$ and $\frac{1}{2}\int_0^1 |\dot{\varphi}_z(s)|^2 ds \le \gamma/2$.

It is then possible to follow the proof of Lemma 1.7, chapter 6 of Freidlin and Wentzell (1979): remark that

$$\mathbb{P}_z^\varepsilon[\tau_{G^c} \le 1] \ge \mathbb{P}\left[\sup_{t \in [0,1]} |Z_t^{\varepsilon,z} - \mathbf{z}_{\varphi_z}(z, t)| \le \delta\right]. \tag{4.10}$$

Second, using that G is a compact set,there exists a convergent sequence (z_k) of G and a sequence (ε_k) such that $\varepsilon_k \to 0$ and such that,

$$\liminf_{\varepsilon \to 0} \inf_{z \in G} \varepsilon^2 \ln \mathbb{P}_z^\varepsilon[\tau_{G^c} \le 1] = \lim_{k \to +\infty} \ln(\mathbb{P}_{z_k}^{\varepsilon_k}[\tau_{G^c} \le 1]).$$

Now, owing to Lemma 3.1 and to (4.10), we deduce that

$$\liminf_{\varepsilon \to 0} \inf_{z \in G} \varepsilon^2 \ln \mathbb{P}_z^\varepsilon[\tau_{G^c} \le 1] \ge -\frac{1}{2}\int_0^1 |\dot{\varphi}_{z_\infty}(s)|^2 ds \ge -\frac{\gamma}{2},$$

where $z_\infty := \lim_{k \to +\infty} z_k$. As a consequence, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$, for all $z \in G$,

$$\mathbb{P}_z^\varepsilon[\tau_{G^c} \le 1] \ge e^{-\gamma\varepsilon^{-2}}$$

and the strong Markov property implies that

$$\forall n \in \mathbb{N} \qquad \mathbb{P}_z^\varepsilon[\tau_{G^c} > n] \le [1 - e^{-\gamma\varepsilon^{-2}}]^n.$$

We obtain for every $z \in G$ and $\varepsilon \in (0, \varepsilon_0]$

$$\mathbb{E}_z^\varepsilon \tau_{G^c} \le \sum_{n=0}^{\infty} [1 - e^{-\gamma\varepsilon^{-2}}]^n \le e^{-\gamma\varepsilon^{-2}}.$$

$\square$

Following the same kind of argument using again the key Lemma 4.1 and the finite time large deviation principle, we also obtain that Lemma 1.8, chapter 6 of Freidlin and Wentzell (1979) still holds. In our context, this leads to the following lemma.

Lemma 4.3 *Assume that* $(\mathbf{H_D})$ *holds and that either* $(\mathbf{H_{Q+}})$ *or* $(\mathbf{H_{Q-}})$ *is satisfied. For any* $i \in \{1 \ldots \ell\}$ *and* $z_i^\star$ *an equilibrium of (2.3), we define for any* $\rho > 0$ *the neighbourhood* $G := B(z_i^\star, \rho)$ *of* $z_i^\star$*. Then, for any* $\gamma > 0$*, there exists* $\delta \in (0, \rho]$ *such that if we define* $g = B(z_i^\star, \delta)$*, we have for* ε *small enough*

$$\inf_{z \in g} \mathbb{E}\left[\int_0^{\tau_{G^c}} \chi_g(Z_t^{\varepsilon,z})dt\right] > e^{-\gamma\varepsilon^{-2}}.$$

4.3.2 Transitions of the Markov chain skeleton

By Proposition 4.2, the idea is now to deduce the behaviour of ν_ε from the control of the transitions of the skeleton chain $(\tilde{Z}_n)_{n\in\mathbb{N}}$. We recall that for any $(\xi_1, \xi_2) \in (\mathbb{R}^d \times \mathbb{R}^d)^2$, $I_t(\xi_1, \xi_2)$ denotes the L^2-minimal cost to go from ξ_1 to ξ_2 in a finite time t:

$$I_t(\xi_1, \xi_2) = \inf_{\varphi \in \mathbb{H}, \mathbf{z}_\varphi(\xi_1, t) = \xi_2} \frac{1}{2}\int_0^t |\dot{\varphi}(s)|^2 ds,$$

and $I(\xi_1, \xi_2) = \inf_{t\geq 0} I_t(\xi_1, \xi_2)$. In the sequel, we will also need to introduce $\tilde{I}(z_i^\star, z_j^\star)$ defined for all $\forall (i,j) \in \{1 \ldots \ell\}^2$ by:

$$\tilde{I}(z_i^\star, z_j^\star) = \inf_{t>0}\left\{\frac{1}{2}\int_0^t |\dot{\varphi}(s)|^2 ds, \varphi \in \mathbb{H}, \mathbf{z}_\varphi(z_i^\star, t) = z_j^\star, \forall s \in [0,t],\ \mathbf{z}_\varphi(z_i^\star, s) \notin \cup_{k\neq i,j} g_k\right\}.$$

The quantity $\tilde{I}(z_i^\star, z_j^\star)$ is the minimal cost to join $z_j^\star$ from $z_i^\star$ avoiding other equilibria of (2.3). In the following proposition, we prove that $\tilde{I}(z_i^\star, z_j^\star)$ is finite.

Proposition 4.3 *For all* $(i,j) \in \{1 \ldots \ell\}^2$, $\tilde{I}(z_i^\star, z_j^\star) < +\infty$.

Proof : In the proof, we assume that $i \neq j$. The idea is to build a controlled trajectory starting at $z_i^\star$ and ending at $z_j^\star$ (in a finite time) that avoids the other equilibria neighbourhoods $\cup_{k\neq(i,j)} g_k$.

We first assume that $d > 1$. In this case, for all fixed $t_0 > 0$, for any ρ_1-neighbourhood g_k of $z_k^\star$, one can find a smooth trajectory $(\mathbf{x}_0(t))_{t\geq 0}$ satisfying $\mathbf{x}_0(0) = x_i^\star$, $\mathbf{x}_0(t_0) = x_j^\star$ and

$$\forall s \in [0; t_0] \qquad \inf_{k\neq i,j} |\mathbf{x}_0(s) - x_k^\star| > \rho_1.$$

Then, denote by $(\mathbf{y}_0(t))_{t\geq 0}$ a solution of $\dot{\mathbf{y}}_0(t) = \nabla U(\mathbf{x}_0(t)) - \mathbf{y}_0(t)$ with initial condition $\mathbf{y}_0(0) = 0$ and let $\varphi_0 \in \mathbb{H}$ satisfy $\dot{\varphi}_0(t) = \dot{\mathbf{x}}_0(t) + \mathbf{y}_0(t)$. We thus obtain a controlled trajectory $\mathbf{z}_{\varphi_0}(z_i^\star, .)$ which satisfies $\mathbf{z}_{\varphi_0}(z_i^\star, t) = (\mathbf{x}_0(t), \mathbf{y}_0(t))$ for all $t \in [0, t_0]$. This way, we have

$$\mathbf{x}_{\varphi_0}(z_i^\star, t_0) = x_j^\star \text{ and } \forall s \in [0; t_0]\ \mathbf{z}_{\varphi_0}(z_i^\star, s) \notin \cup_{k\neq i,j} g_k.$$

It remains now to join $(x_j^\star, 0)$ from $(x_j^\star, y_0(t_0))$ without hitting $\cup_{k\neq i,j} g_k$. Let $(\mathbf{x}_1(t), \mathbf{y}_1(t))_{t\geq t_0}$ be defined for all $t \geq t_0$ by $\mathbf{x}_1(t) = x_j^\star$ and $\mathbf{y}_1(t) = \mathbf{y}_0(t_0)e^{t_0 - t}$ (so that $\mathbf{y}_1$ is a solution of $\dot{\mathbf{y}_1} = -\mathbf{y}_1$ with $\mathbf{y}_1(t_0) = \mathbf{y}_0(t_0)$). Once again, $(\mathbf{x}_1(t), \mathbf{y}_1(t))_{t\geq t_0}$ can be viewed as a controlled trajectory $\mathbf{z}_{\varphi_1}((x_j^\star, \mathbf{y}_0(t_0), .)$ by setting $\dot{\varphi}_1(t) = \mathbf{y}_1(t)$.

Furthermore, $\mathbf{z}_{\varphi_1}((x_j^\star, \mathbf{y}_0(t_0)), t) \xrightarrow{t\to+\infty} (x_j^\star, 0)$. As a consequence, there exists T such that $\mathbf{z}_{\varphi_1}((x_j^\star, \mathbf{y}_0(t_0)), T) \in g_j$. Hence, one can find a controlled trajectory starting from z_i and ending into any sufficiently small neighbourhood of z_j in a finite time and avoids the other ρ_1 neighbourhood of $\cup_{k\neq(i,j)} g_k$. It remains to use Lemma 4.1 to obtain a controlled trajectory starting at $\mathbf{z}_{\varphi_1}((x_j^\star, \mathbf{y}_0(t_0)), T)$ and ending at point $z_j^\star$ within a finite time. The global controlled trajectory initialized at $z_i^\star$ ends at $z_j^\star$ with a finite L^2 control cost. The result then follows when $d > 1$.

Consider now the case $d = 1$ and let $x_i^\star, x_j^\star$ be two critical points of U. Without loss of generality, one may suppose that $x_i^\star < x_j^\star$. From $(\mathbf{H_D})$, the number of critical points which belong to $[x_i^\star, x_j^\star]$ is finite (denoted by p):

$$x_i^\star < x_{i_1}^\star < \cdots < x_{i_p}^\star < x_{i_{p+1}}^\star := x_j^\star.$$

Now, we consider a path which joins $x_i^\star$ to $x_j^\star$ parametrised as

$$\mathbf{x}_\alpha(t) = x_i^\star + \alpha(t)[x_j^\star - x_i^\star],$$

with $\alpha(0) = 0$ and $\alpha(T) = 1$ for T large enough which will be given later. Of course, $\mathbf{y}_\alpha(t)$ is then defined as

$$\forall t \in [0;T] \qquad \mathbf{y}_\alpha(t) = \int_0^t e^{s-t} U'\left(x_i^\star + \alpha(s)[x_j^\star - x_i^\star]\right) ds. \tag{4.11}$$

For the sake of simplicity, we consider only increasing maps α. If $p = 0$, we know that $(\mathbf{x}_\alpha(t), \mathbf{y}_\alpha(t))_{t\in[0;1]}$ avoids $\cup_{k\neq(i,j)}(x_k^\star, 0)$ and then $\tilde{I}(z_i^\star, z_j^\star) < +\infty$ which proves the proposition. If $p > 0$, there exists $t_1, \ldots t_p$ such that $\mathbf{x}_\alpha(t_k) = x_{i_k}^\star$ and we shall prove that one can find α such that $\mathbf{y}_\alpha(t_k) \neq 0$. Since we consider only increasing paths, we first show that one can find a monotone α such that $\mathbf{y}_\alpha(t_1) \neq 0$. Let α be any $\mathcal{C}^1$ increasing parametrisation defined on $[0; t_1]$. We know that U' does not vanish on $]x_i^\star, x_{i_1}^\star[$ and from equation (4.11), $\mathbf{y}_\alpha(t_1) \neq 0$. Suppose without loss of generality that $\mathbf{y}_\alpha(t_1) < 0$, which means that $U' < 0$ on $]x_i^\star, x_{i_1}^\star[$. Since we know that $U"(x_{i_1}^\star) \neq 0$, one can find $\delta > 0$ small enough such that $U' > 0$ on $]x_{i_1}^\star; x_{i_1}^\star + \delta[$. Let $\xi_1 \in]x_{i_1}^\star; x_{i_1}^\star + \delta[$, we continue the parametrisation α from t_1 to $\tilde{t}_1$ such that $\mathbf{x}(\tilde{t}_1) = \xi_1$ and α remains constant on $[\tilde{t}_1; \tilde{t}_1 + \delta t_1]$. Expanding the integral that defines $\mathbf{y}_\alpha$ (see Equation (4.11)) between $[0, t_1], [t_1, \tilde{t}_1]$ and $[\tilde{t}_1, \tilde{t}_1 + \delta t_1]$, simple computation yields

$$\mathbf{y}_\alpha(\tilde{t}_1 + \delta t_1) = \mathbf{y}_\alpha(t_1) e^{t_1 - \tilde{t}_1 - \delta t_1} + \int_{t_1}^{\tilde{t}_1} e^{s - \tilde{t}_1 + \delta t_1} U'(x_i^\star + \alpha(s)[x_j^\star - x_i^\star]) ds + U'(\xi_1)[1 - e^{-\delta t_1}].$$

Hence, it is obvious to see that we can find a sufficiently large δt_1 such that $y_\alpha(\tilde{t}_1 + \delta t_1) > 0$ since $U'(\xi_1) > 0$. We continue the parametrisation α until t_2, time at which $x_{i_2}^\star$ is reached. By construction, $\mathbf{y}_\alpha(t_2) > 0$. Now, one can repeat the same argument by induction to find α such that $\mathbf{y}_\alpha(t_k) \neq 0$ for all $k \leq p + 1$ such that $\mathbf{x}_\alpha(t_k) = x_{i_k}^\star$.

Thus, at time t_{p+1}, $\mathbf{x}_\alpha(t_{p+1}) = x_j^\star$ and $\mathbf{y}_\alpha(t_{p+1}) \neq 0$. It remains now to join $z_j^\star = (x_j^\star, 0)$ without hitting $\cup_{k\neq i,j} g_k$ and this can be achieved exactly as in dimension $d > 1$. □

It is now possible to compute the estimations of the invariant measure $\tilde{\mu}_\varepsilon^{\partial g}$ of the skeleton chain. The key estimation of the transition probability of $(\tilde{Z}_n)_{n\in\mathbb{N}}$ denoted $\tilde{P}^\varepsilon(z, .)$ is as follows.

Proposition 4.4 *For any $\gamma > 0$, there exists a sufficiently small ρ_0 and ρ_1 satisfying $0 < \rho_1 < \rho_0$ such that with the definition (4.2) and (4.4), we have for ε small enough*

$$\forall (i,j) \in \{1 \ldots \ell\}^2 \quad \forall z \in \partial g_i \qquad e^{-\varepsilon^{-2}[\tilde{I}(z_i^\star, z_j^\star) + \gamma]} \leq \tilde{P}^\varepsilon(z, \partial g_j) \leq e^{-\varepsilon^{-2}[\tilde{I}(z_i^\star, z_j^\star) - \gamma]}.$$

The proof is a simple adaptation of the proof of Lemma 2.1 and 2.2, chapter 6 of (Freidlin & Wentzell, 1979) in view of our three Lemmas 4.1, 4.2, 4.3 and of Proposition 4.3.

4.4 $\{i\}$-Graphs and invariant measure estimation

We recall that $\{i\}$-Graphs for Markov chains are defined in paragraph 2.4.2. for any finite set $\{z_1^\star, \ldots, z_\ell^\star\}$ and the set of all possible $\{i\}$-Graphs is referred as $\mathcal{G}(i)$. According to this definition, we can set

$$\mathcal{W}(z_i^\star) = \min_{\mathcal{IG}\in\mathcal{G}(i)} \sum_{(m\to n)\in\mathcal{IG}} \tilde{I}(z_m^\star, z_n^\star),$$

and as pointed in Lemma 4.1 of Freidlin and Wentzell (1979), one can check that

$$\mathcal{W}(z_i^\star) = \min_{\mathcal{IG}\in\mathcal{G}(i)} \sum_{(m\to n)\in\mathcal{IG}} I(z_m^\star, z_n^\star).$$

We are now able to obtain the main result of this paragraph. From the skeleton representation (Proposition 4.2) and from the estimations given by Lemma 4.3 and Proposition 4.4, one may describe the asymptotic behaviour of ν_ε as $\varepsilon \longrightarrow 0$. The result is as follows.

Theorem 4.1 *For any $\gamma > 0$, there exists ρ_1 satisfying $0 < \rho_1 < \rho_0$ such that if $g_j = B(z_j^\star, \rho_1)$:*

$$e^{-\varepsilon^{-2}\left[\mathcal{W}(z_i^\star) - \min_{j\in\{1,\dots,\ell\}} \mathcal{W}(z_j^\star) + \gamma\right]} \leq \nu_\varepsilon(g_j) \leq e^{-\varepsilon^{-2}\left[\mathcal{W}(z_i^\star) - \min_{j\in\{1,\dots,\ell\}} \mathcal{W}(z_j^\star) - \gamma\right]},$$

for all $i \in \{1\dots\ell\}$. As well, in terms of W, we get that

$$e^{-\varepsilon^{-2}\left[W(z_i^\star) + \gamma\right]} \leq \nu_\varepsilon(g_j) \leq e^{-\varepsilon^{-2}\left[W(z_i^\star) - \gamma\right]}, \quad \forall i \in \{1\dots\ell\}.$$

The proof of this theorem is straightforward according to the previous results: the invariant measure ν_ε only weights small neighbourhoods of global minima of W, when $\varepsilon \to 0$. Such global minima are appropriately described using the quasipotential I and the function $\mathcal{W}$ obtained through the $\{i\}$-graph structures.

5 Minoration and Majoration of the rate function with a double-well landscape in $\mathbb{R}$

This last part is devoted to the proof of Theorem 2.3. Here, we concentrate on a one dimensional potential U with a double-well profile and on the memory gradient system with fixed memory parameter λ. In this case we obtain the precise behaviour of our measure ν_ε when $\varepsilon \to 0$. From Freidlin and Wentzell estimates, we can see that ν_ε concentrates on the minima of W (this set is also the minima of $\mathcal{W}$). Here we want to show that ν_ε concentrates on the global minimum of U. To this end, we consider of a double-well potential U whose two minima are denoted by $x_1^\star$ and $x_2^\star$. In order to show the final result, one needs to compare the costs $I(z_1^\star, z_2^\star)$ and $I(z_2^\star, z_1^\star)$ for the two stable equilibria $z_1^\star = (x_1^\star, 0)$ or $z_2^\star = (x_2^\star, 0)$. Without loss of generality, we fix $x_1^\star < x_2^\star$ and assume that there exists a uniquelocal maximum $x^\star$ of U such that $x_1^\star < x^\star < x_2^\star$ and $U'(x^\star) = 0, U"(x^\star) < 0$. We assume moreover that $U(x_1^\star) < U(x_2^\star)$. Such potential U is represented in Figure 2.

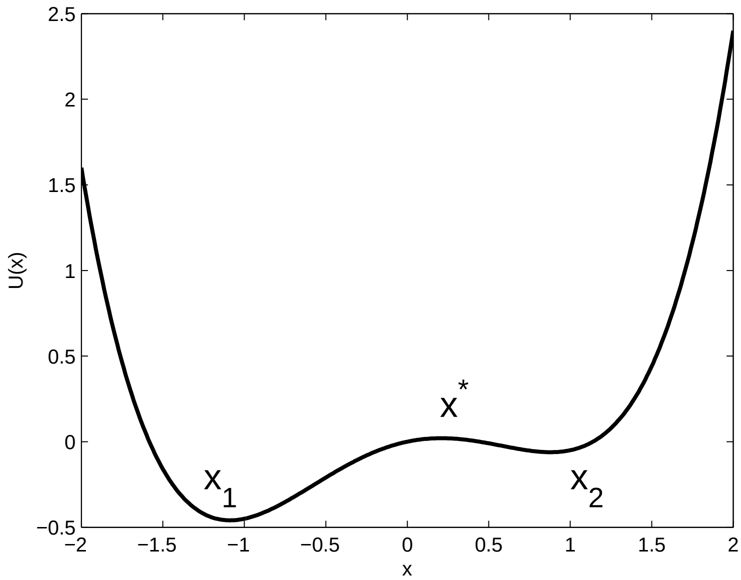

Figure 2: Example of double-well potential (here $U(x) = x^4/4 - x^2/2 + x/5$).

We first describe how one can provide a lower bound of the cost $I(z_1^\star, z_2^\star)$. We propose two approaches using sharp estimates of particular Lyapunov functions. In the next subsection, we adopt a non-degenerate approach where the main idea is to project the drift vector field onto the gradient of the Lyapunov function. However, even if the idea seems to be original, the bounds are not very satisfactory (see Proposition 5.1). In Subsection 5.2 we propose a second approach which provides better bounds (see Proposition (5.2)).

5.1 Minoration using a non-degenerate approach

Let $(\beta, \gamma) \in \mathbb{R}^2$, in this section, we consider the following Lyapunov function defined by

$$\mathcal{L}_{\beta,\gamma}(z) := \mathcal{L}_{\beta,\gamma}(x, y) = U(x) + \beta y^2/2 - \gamma U'(x)y.$$

For the sake of simplicity, we will omit the dependence on β and γ and denote by $\mathcal{L}$ this function. Here, the main idea relies on the fact that $\nabla\mathcal{L}$ corresponds to a favoured direction of the drift b. This will allow us to control the L^2 cost to move the system from $z_1^\star$ to $z_2^\star$. First let us remark that the cost I is necessarily bounded from below by the L^2 cost for an elliptic system. In particular, in the elliptic context the L^2 cost is defined as

$$I_{\mathcal{E},T}(z_1^\star, z_2^\star) = \inf_{\varphi\in\mathbb{H}} \left\{ \frac{1}{2}\int_0^T |\dot\varphi_1 - b_1(\varphi)|^2 + |\dot\varphi_2 - b_2(\varphi)|^2 \quad | \quad \varphi(0) = z_1^\star, \varphi(T) = z_2^\star \right\},$$

which can also been written as

$$I_{\mathcal{E},T}(z_1^\star, z_2^\star) = \inf_{(u,v)\in\mathbb{L}^2([0;T])} \left\{ \frac{1}{2}\int_0^T u^2(s) + v^2(s)ds \quad | \quad \dot{\mathbf{z}} = b(\mathbf{z}) + \begin{pmatrix} u \\ v \end{pmatrix}, \quad \mathbf{z}(0) = z_1^\star, \mathbf{z}(T) = z_2^\star \right\}. \tag{5.1}$$

As a consequence, since the set of admissible control for the degenerate cost I_T is contained in the set of admissible controls for $I_{\mathcal{E},T}$ (v is forced to be 0 in Equation (5.1)), we easily deduce that I_T is greater than $I_{\mathcal{E},T}$. This way, a lower bound of $I_{\mathcal{E},T}$ will yield a lower bound for I_T.

Now, let u and v be admissible controls for $I_{\mathcal{E},T}$, we have

$$u^2 + v^2 = |\dot{\mathbf{z}} - b(\mathbf{z})|^2. \tag{5.2}$$

Adapting the approach developed in Chiang et al. (1987), we shall use the Lyapunov function $\mathcal{L}$ to bound from below the term above (somehow the Lyapunov function $\mathcal{L}$ will play the role of U). Indeed, if $\nabla\mathcal{L} \neq 0$, one can decompose b as follows

$$b(\mathbf{z}) = b_{\nabla\mathcal{L}(\mathbf{z})} + b_{\nabla\mathcal{L}(\mathbf{z})^\perp}, \tag{5.3}$$

where $b_{\nabla\mathcal{L}(\mathbf{z})}$ is the orthogonal projection of b on the direction $\nabla\mathcal{L}$. In the special case $\nabla\mathcal{L} = 0$, we fix $b_{\nabla\mathcal{L}(\mathbf{z})} = 0$ so that Equation (5.3) makes sense for any $\mathbf{z}$. Let us now remark that

$$\begin{aligned} |\dot{\mathbf{z}} - b(\mathbf{z})|^2 &= |\dot{\mathbf{z}} - b_{\nabla\mathcal{L}(\mathbf{z})} - b_{\nabla\mathcal{L}(\mathbf{z})^\perp}|^2 \\ &= |\dot{\mathbf{z}} - b_{\nabla\mathcal{L}(\mathbf{z})^\perp}|^2 + |b_{\nabla\mathcal{L}(\mathbf{z})}|^2 - 2\langle\dot{\mathbf{z}}; b_{\nabla\mathcal{L}(\mathbf{z})}\rangle \\ &\geq -2\frac{\langle b(\mathbf{z}); \nabla\mathcal{L}(\mathbf{z})\rangle}{|\nabla\mathcal{L}(\mathbf{z})|^2}\langle\dot{\mathbf{z}}; \nabla\mathcal{L}(\mathbf{z})\rangle. \end{aligned}$$

This way, if one can find β and γ such that there exists $\alpha > 0$ such that

$$\forall z \in \mathbb{R}^2 \qquad -\frac{\langle b(z); \nabla\mathcal{L}(z)\rangle}{|\nabla\mathcal{L}(z)|^2} \geq \alpha, \tag{5.4}$$

then it is possible to conclude that for all $T > 0$

$$\begin{aligned} I_{\mathcal{E},T}(z_1, z_2) &= \inf_{(u,v)\in\mathbb{L}^2([0;T])} \left\{ \frac{1}{2}\int_0^T u^2(s) + v^2(s)ds, \ \dot{\mathbf{z}} = b(\mathbf{z}) + \begin{pmatrix} u \\ v \end{pmatrix}, z(0) = z_1, z(T) = z_2 \right\} \\ &\geq \inf_\varphi \left\{ \int_0^T -\frac{\langle b(\mathbf{z}(s)), \nabla\mathcal{L}(\mathbf{z}(s))\rangle}{|\nabla\mathcal{L}(\mathbf{z}(s))|^2}\langle\dot{\mathbf{z}}(s), \ \nabla\mathcal{L}(\mathbf{z}(s))\rangle ds, \varphi(0) = z_1, \varphi(T) = z_2 \right\}, \\ &\geq \alpha[\mathcal{L}(\mathbf{z}(t)) - \mathcal{L}(z_1^\star)], \qquad \forall t \in [0, T]. \end{aligned}$$

Now, remark that for admissible controls, $(\mathbf{z}(t))_{t\geq 0}$ moves continuously from z_1 to z_2 and there exists $t^\star$ such that $\mathbf{x}(t^\star) = x^\star$. We then obtain

$$I_{\mathcal{E},T}(z_1^\star, z_2^\star) \geq \alpha[\mathcal{L}(\mathbf{z}(t^\star)) - \mathcal{L}(z_1^\star)].$$

In the definition of $\mathcal{L}$, if $\beta \geq 0$, one obtains a lower bound of the cost of the form

$$I_{\mathcal{E},T}(z_1^\star, z_2^\star) \geq \alpha[U(x^\star) - U(x_1^\star)].$$

If $\beta \leq 0$, the only available minoration is obtained taking $t = T$ and we then get the weaker bound

$$I_{\mathcal{E},T}(z_1, z_2) \geq \alpha[U(x_2^\star) - U(x_1^\star)].$$

The next proposition provides a lower bound of the cost in the (restrictive) case of subquadratic potential U (that is under H_{Q-}).

Proposition 5.1 *Suppose $U \in \mathcal{C}^2(\mathbb{R}, \mathbb{R})$ and define $M = \|U"\|_\infty$, then*

$$I_{\mathcal{E}}(z_1^\star, z_2^\star) \geq \alpha_\lambda(M)[U(x^\star) - U(x_1^\star)],$$

where $\alpha_\lambda(M)$ satisfies the asymptotic properties

$$\lim_{M\to 0} \alpha(M) = \frac{1}{1 + \frac{1}{2\lambda^2} + \sqrt{1 + \frac{1}{4\lambda^4}}}, \qquad \alpha(M) \sim_{M\to\infty} \frac{\lambda}{M\left[1 + \frac{1}{2\lambda^2} + \sqrt{1 + \frac{1}{4\lambda^4}}\right]}.$$

At last, we have

$$\forall M > 0 \qquad \lim_{\lambda\mapsto+\infty} \alpha_\lambda(M) = 2.$$

<u>*Proof :*</u> The idea is to optimize the ratio given in Equation (5.4) for the largest possible α. Such ratio can be written as a quadratic form on y and $U'(x)$. This way, an algebraic argument based on a simultaneous reduction of these quadratic forms allows to obtain a suitable calibration for M and α.

Let us first compute the projection of $b(z)$ on $\nabla\mathcal{L}(z)$ when it does not vanish. We can expand $\langle b(z); \nabla\mathcal{L}(z)\rangle$ as a quadratic form on variables $(U'(x), y)$.

$$\begin{aligned}
\langle b(z); \nabla\mathcal{L}(z)\rangle &= \langle\begin{pmatrix} -y \\ \lambda[U'(x) - y]\end{pmatrix}; \begin{pmatrix} U'(x) - \gamma U"(x)y \\ \beta y - \gamma U'(x)\end{pmatrix}\rangle \\
&= -[\beta\lambda - \gamma U"(x)]y^2 - \gamma\lambda U'(x)^2 + yU'(x)[\gamma\lambda + \beta\lambda - 1]
\end{aligned}$$

If we set

$$M_1 = \begin{pmatrix} \gamma\lambda & -\frac{\gamma\lambda+\beta\lambda-1}{2} \\ -\frac{\gamma\lambda+\beta\lambda-1}{2} & [\beta\lambda - \gamma U"(x)]\end{pmatrix},$$

we then obtain

$$\langle b(z); \nabla\mathcal{L}(z)\rangle = -\begin{pmatrix} U'(x) \\ y\end{pmatrix}^t M_1 \begin{pmatrix} U'(x) \\ y\end{pmatrix}. \tag{5.5}$$

In the same way, one can compute that

$$|\nabla\mathcal{L}(z)|^2 = U'(x)^2(1 + \gamma^2) + y^2(\gamma^2 U"(x)^2 + \beta^2) - 2U'(x)y(\gamma U"(x) + \beta\gamma),$$

so that

$$|\nabla\mathcal{L}(z)|^2 = \begin{pmatrix} U'(x) \\ y\end{pmatrix}^t M_2 \begin{pmatrix} U'(x) \\ y\end{pmatrix}, \tag{5.6}$$

where

$$M_2 = \begin{pmatrix} 1 + \gamma^2 & -\gamma U"(x) - \beta\gamma \\ -\gamma U"(x) - \beta\gamma & \beta^2 + \gamma^2 U"(x)^2\end{pmatrix}.$$

In order to have an equal equilibrium between the repelling effect on $U'(x)^2$ and y^2 on the quadratic form defined by M_1, a natural choice for β and γ would be

$$\begin{cases} \min_{x\in\mathbb{R}}[\beta\lambda - \gamma U"(x)] = \gamma\lambda \\ 1-\gamma\lambda-\beta\lambda \quad = 0 \end{cases}$$

Hence, we set

$$\beta = \frac{\lambda+M}{\lambda(2\lambda+M)} \qquad \text{and} \qquad \gamma = \frac{1}{2\lambda+M}.$$

The end of the proof then falls into an algebraic argument : denote $(a,b) = (U'(x), y)$, we are looking for a bound similar to (5.4) with the larger possible α . The projection of b on $\nabla\mathcal{L}$ can be expressed as

$$-\frac{\langle b(z); \nabla\mathcal{L}(z)\rangle}{|\nabla\mathcal{L}(z)|^2} = \frac{q_{M_1}(a,b)}{q_{M_2}(a,b)},$$

where q_{M_1} and q_{M_2} are the two quadratic forms defined from expressions (5.5) and (5.6). To bound the ratios of these two quadratic forms, remark that M_1 is invertible except if $M = 0$ which is a rather trivial case. Then, M_1 is symetric and positive definite as well as M_2 is non-negative and symetric. It is possible to use a simultaneous reduction of q_{M_1} and q_{M_2}. We denote ρ_1 and ρ_2 the eigenvalues of $M_1^{-1}M_2$ associated to eigenvectors e_1, e_2 which are an orthonormal basis for q_{M_1}, if $(\tilde{a}, \tilde{b})$ are the coordinates in this basis we have

$$\frac{q_{M_1}(a,b)}{q_{M_2}(a,b)} = \frac{\tilde{a}^2 + \tilde{b}^2}{\rho_1\tilde{a}^2 + \rho_2\tilde{b}^2},$$

and the minimum of q_{M_1}/q_{M_2} is then

$$\min_{(a,b)} \frac{q_{M_1}(a,b)}{q_{M_2}(a,b)} = \frac{1}{\rho_1} \wedge \frac{1}{\rho_2} = \frac{1}{\rho_1 \vee \rho_2}.$$

With our choice of β and γ and setting $\xi = U"(x)/M \in [-1;1]$, simple algebra yields

$$M_1^{-1}M_2(\xi) = \begin{pmatrix} 2 + \frac{M}{\lambda} + \frac{1}{\lambda(2\lambda+M)} & -\left(\frac{M}{\lambda}\xi + \frac{1+\frac{M}{\lambda}}{\lambda(2\lambda+M)}\right) \\ -\left(\frac{M\xi + \frac{\lambda+M}{\lambda(2\lambda+M)}}{\lambda+M(1-\xi)}\right) & \frac{\xi^2M^2 + (1+M/\lambda)^2}{(\lambda+M(1-\xi))(2\lambda+M)} \end{pmatrix}.$$

For small M, it is immediate to show that $M_1^{-1}M_2$ becomes independent of ξ and that

$$\frac{1}{\rho_1 \vee \rho_2} = \frac{1}{1 + \frac{1}{2\lambda^2} + \sqrt{1 + \frac{1}{4\lambda^4}}}.$$

For large M, the maximum eigenvalue of $M_1^{-1}M_2$ is reached for $\xi = 1$ and one obtains again after tedious computations that

$$\frac{1}{\rho_1 \vee \rho_2} = \frac{\lambda}{M\left[1 + \frac{1}{2\lambda^2} + \sqrt{1 + \frac{1}{4\lambda^4}}\right]}.$$

For any $M > 0$, the coefficient $\alpha(M)$ is then obtained by

$$\alpha(M) = \min_{\xi\in[-1;1], \rho\in \mathrm{Sp}(M_1^{-1}M_2(\xi))} \frac{1}{\rho}$$

and the result holds.

At last, remark that for any large λ, the matrix $M_1^{-1}M_2$ becomes diagonal with the two eigenvalues 2 and 0. This proves that $\lim_{\lambda\to+\infty} \alpha_\lambda(M) = 2$. □

5.2 Minoration using a degenerate approach (Proof of (ii) of Theorem 2.3)

In order to take into account the degeneracy of the dynamical system for the control cost, we directly bound the terms in the integral of $I_T(z_1, z_2)$ by a gradient of a suitable Lyapunov function. This may lead to better estimates since obviously in the previous paragraph we use a minoration technique based on elliptic argument.

Let $\alpha > 0$ and $(\beta, \gamma) \in \mathbb{R}^2$, here we consider the Lyapunov function defined by

$$\mathcal{L}_{\alpha,\beta,\gamma}(z) = \mathcal{L}_{\alpha,\beta,\gamma}(x, y) := \alpha U(x) + \beta y^2/2 - \gamma y U'(x).$$

We are looking for an ideal choice of (α, β, γ). For all $\varphi \in \mathbb{H}(\mathbb{R}_+, \mathbb{R}^d)$, we set $u = \dot{\varphi}$. If u denotes any admissible control and $(\mathbf{z}(t))_{t\geq 0}$ the corresponding controlled trajectory, we aim to obtain a bound such as for all $\varphi \in \mathbb{H}(\mathbb{R}_+, \mathbb{R}^d)$, we have

$$\forall t \geq 0 \qquad u^2(t) \geq 2\frac{\mathrm{d}\mathcal{L}_{\alpha,\beta,\gamma}(\mathbf{z}(t))}{\mathrm{d}t}. \tag{5.7}$$

Recall that $t^\star$ is the first time such that x reaches the local maximum of U (*i.e.* $\mathbf{x}(t^\star) = x^\star$). Such lower bound is usefull especially if α is positive, largest as possible and β non-negative. Indeed assume that we can obtain lower bound of the form (5.7), we then have for all T

$$\begin{aligned}
I_T(z_1^\star, z_2^\star) &= \inf_{u\in\mathbb{L}^2([0;T])} \left\{ \frac{1}{2}\int_0^T u^2(s)ds,\ \dot{\mathbf{z}} = b(\mathbf{z}) + \begin{pmatrix} u \\ 0 \end{pmatrix}, \mathbf{z}(0) = z_1^\star, \mathbf{z}(T) = z_2^\star \right\} \\
&\geq \inf_{u\in\mathbb{L}^2([0;T])} \left\{ \frac{1}{2}\int_0^{t^\star} u^2(s)ds,\ \dot{\mathbf{z}} = b(\mathbf{z}) + \begin{pmatrix} u \\ 0 \end{pmatrix}, \mathbf{z}(0) = z_1^\star, \mathbf{z}(T) = z_2^\star \right\} \\
&\geq \alpha[U(x^\star) - U(x_1^\star)] + \frac{\beta}{2}\mathbf{y}(t^\star)^2 \\
&\geq \alpha[U(x^\star) - U(x_1^\star)].
\end{aligned} \tag{5.8}$$

The next proposition shows that indeed such minoration (5.7) holds for some suitable choice of β, γ. In some case, this minoration is almost optimal.

Proposition 5.2 *For all $\alpha \in [0; 2[$, there exist some explicit constants $m_\lambda(\alpha), \beta^\star(\alpha), \gamma^\star(\alpha)$ such that (5.7) is true for $\beta = \beta^\star(\alpha), \gamma = \gamma^\star(\alpha)$ and for all one-dimensional double well potential U satisfying $\|U"\|_\infty = M < m_\lambda(\alpha)$. In this case, we get*

$$I(z_1^\star, z_2^\star) \geq \alpha\Big[U(x^\star) - U(x_1^\star)\Big].$$

Proof : In order to obtain a minoration such as (5.7), we fix any $t > 0$ and any admissible control u. Dropping the time parameter, note that, we have

$$u^2 = |\dot{\mathbf{x}} + \mathbf{y}|^2 = \dot{\mathbf{x}}^2 + \mathbf{y}^2 + 2\dot{\mathbf{x}}\mathbf{y},$$

and since $\dot{\mathbf{y}} = \lambda[U'(\mathbf{x}) - \mathbf{y}]$, one can also compute

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{L}_{\alpha,\beta,\gamma}(\mathbf{z})}{\mathrm{d}t} &= \alpha\dot{\mathbf{x}}U'(\mathbf{x}) + \lambda\beta\mathbf{y}(U'(\mathbf{x}) - \mathbf{y}) - \gamma\mathbf{y}\dot{\mathbf{x}}U"(\mathbf{x}) - \lambda\gamma(U'(\mathbf{x}) - \mathbf{y})U'(\mathbf{x}) \\
&= -\lambda\beta\mathbf{y}^2 - \lambda\gamma U'(\mathbf{x})^2 + \alpha\dot{\mathbf{x}}U'(\mathbf{x}) + \lambda\mathbf{y}U'(\mathbf{x})(\beta + \gamma) + \dot{\mathbf{x}}\mathbf{y}(-\gamma U"(\mathbf{x})).
\end{aligned}$$

Let us now define M_1 and M_2 as

$$M_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 0 & \frac{\alpha}{2} & -\frac{\gamma}{2}U"(x) \\ \frac{\alpha}{2} & -\lambda\gamma & \frac{\beta+\gamma}{2}\lambda \\ -\frac{\gamma}{2}U"(x) & \frac{\beta+\gamma}{2}\lambda & -\beta\lambda \end{pmatrix},$$

for all $x \in \mathbb{R}$. This way, we can write

$$u^2 = (\dot{\mathbf{x}}, U'(\mathbf{x}), \mathbf{y}) M_1 (\dot{\mathbf{x}}, U'(\mathbf{x}), \mathbf{y})^t, \qquad \text{and} \qquad \frac{\mathrm{d}\mathcal{L}_{\alpha,\beta,\gamma}(\mathbf{z})}{\mathrm{d}t} = (\dot{\mathbf{x}}, U'(\mathbf{x}), \mathbf{y}) M_2 (\dot{\mathbf{x}}, U'(\mathbf{x}), \mathbf{y})^t.$$

Remark that again, the product $yU'(x)$ is essential in the structure of the Lyapunov function since it creates some repelling effect in M_2 in variables y^2 and $U'(x)^2$. Without this term, there is no chance to obtain positiveness of $M_1 - 2M_2$. Moreover, we immediately get that β and γ should be positives.

It is then sufficient to obtain that the symmetric matrix $S := M_1 - 2M_2$ is positive. We again introduce a parameter ξ but this time it is easier to manipulate $\xi = U"(x) \in [-M; M]$. The matrix S becomes:

$$S = \begin{pmatrix} 1 & -\alpha & 1+\gamma\xi \\ -\alpha & 2\gamma\lambda & -(\beta+\gamma)\lambda \\ 1+\gamma\xi & -(\beta+\gamma)\lambda & 1+2\beta\lambda \end{pmatrix}.$$

S is positive if and only if principal minors are positives.

First and Second minors We trivially have $\Delta_1 = 1 > 0$. Hence, we compute

$$\Delta_2 := \det \begin{pmatrix} 1 & -\alpha \\ -\alpha & 2\gamma\lambda \end{pmatrix} = 2\gamma\lambda - \alpha^2.$$

Hence, Δ_2 is positive as soon as

$$\gamma > \frac{\alpha^2}{2\lambda}. \tag{5.9}$$

Third minor Regarding now $\Delta_3 := \det S$, after several computations, we obtain

$$\Delta_3(\xi) \quad = \quad C + B\xi - A\xi^2$$

where

$$A = 2\lambda\gamma^3, B = 2\alpha\gamma\lambda(\beta+\gamma) - 4\lambda\gamma^2, \quad \text{and} \quad C = (1+2\beta\lambda)\left(2\lambda\gamma - \alpha^2\right) - \lambda^2(\beta+\gamma)^2 + 2\alpha\lambda(\beta+\gamma) - 2\gamma\lambda.$$

Hence, Δ_3 is a quadratic polynomial of ξ, it is impossible to obtain the positiveness of $\Delta_3(\xi)$ for all $\xi \in \mathbb{R}$ (this justifies we suppose that $\|U"\|_\infty < +\infty$). For any $\alpha > 0$, we aim to maximise the absolute values of the roots of Δ_3 among the convenient choices of β and γ. By a symmetry argument, it is easy to check that one should have necessarily $B = 0$ since in this case the roots of Δ_3 are opposite.

Thus the parameter β can be expressed in terms of α and γ:

$$\beta = \beta(\alpha, \gamma) = \gamma\left(2/\alpha - 1\right),$$

and for this choice, the roots of Δ_3 are

$$x_\pm(\alpha, \gamma) = \pm\sqrt{\frac{-2(1/\alpha - 1)^2\lambda}{\gamma} + \frac{1 + (\alpha-1)^2}{\gamma^2} - \frac{\alpha^2}{2\lambda\gamma^3}}.$$

Note also that for this choice, we obtain

$$\Delta_3(0) = -\alpha^2 + 2\lambda\gamma(1 + (\alpha-1)^2) - \gamma^2(4(1/\alpha - 1)^2)\lambda^2.$$

Since $\Delta_3(0)$ must be positive, it is easy to show that $\alpha < 2$ (when $\alpha = 2$, $\Delta_3(0) = -(\gamma\lambda - 2)^2$ which is the limiting case). It is possible to maximise $|x_\pm(\alpha, \gamma)|$ with respect to γ for $\alpha \in [0; 2[$. Differentiating with respect to γ, the optimal $\gamma(\alpha)$ is solution of

$$\lambda(1/\alpha - 1)^2\gamma^2 - [1 + (\alpha-1)^2]\gamma + 3\alpha^2/(4\lambda) = 0.$$

Solving this equation, we then successively obtain

$$\gamma_\lambda^\star(\alpha) = \frac{[1+(\alpha-1)^2] - \sqrt{[1+(\alpha-1)^2]^2 - 3\alpha^2(1/\alpha-1)^2}}{2(1/\alpha-1)^2\lambda}, \tag{5.10}$$

and

$$\beta^\star(\alpha) = \beta(\alpha, \gamma_\lambda^\star(\alpha)).$$

Compatibility of (5.9) **and** (5.10) It remains now to check that our chosen value $\gamma_\lambda^\star(\alpha)$ satisfies (5.9). In this view, a short variational study of $\gamma \mapsto \psi_{\lambda,\alpha}(\gamma) := \lambda(1/\alpha-1)^2\gamma^2 - [1+(\alpha-1)^2]\gamma + 3\alpha^2/(4\lambda)$ shows that $\psi_{\lambda,\alpha}$ reaches its minimum at $\frac{\alpha^2}{2\lambda} \times \frac{2-2\alpha+\alpha^2}{(1-\alpha)^2} > \frac{\alpha^2}{2\lambda}$. Moreover, a simple computation shows that

$$\psi_{\lambda,\alpha}(\frac{\alpha^2}{2\lambda}) = \frac{\alpha^2}{4\lambda}(2+2\alpha-\alpha^2) > 0, \quad \forall \alpha \in [0,2].$$

We immediately deduce that $\gamma_\lambda^\star(\alpha) > \frac{\alpha^2}{4\lambda}$ and (5.10) implies (5.9).

Synthesis The maximum admissible value for $\|U"\|_\infty$ defined as $m_\lambda(\alpha)$ in this proposition is then obtained using $\gamma^\star(\alpha)$ in $x_\pm(\alpha,\gamma)$, that is,

$$m_\lambda(\alpha) = x_+(\alpha, \gamma_\lambda^\star(\alpha)).$$

Now since (5.8) is true for all T the result holds for the cost I. □

When M is large, the admissible values for α vanish and our lower bound becomes useless. When $M \mapsto 0$, we obtain $I(z_1, z_2) \geq 2[U(x^\star) - U(x_1)]$ which is optimal in view of the upper bound constructed in the next paragraph (it is obviously better than the bound obtained in Proposition 5.1). The evolution of admissible α is shown in Figures 3 for several values of λ.

As announced in the beginning of the section, one may remark that the second approach is clearly more efficient than the first one. However, we chose to keep the first approach since the idea can be of interest in a more general context, especially in an elliptic case with a drift vector field which is not a gradient.

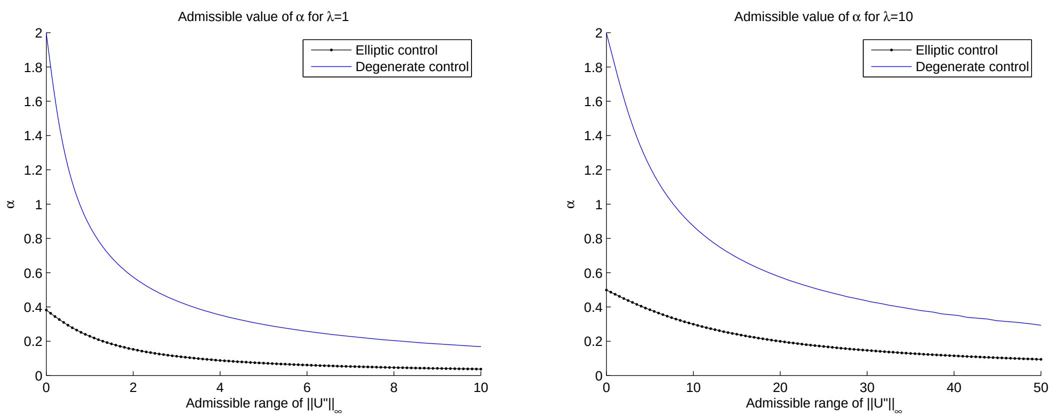

Figure 3: Evolution of the maximum size of α with $\|U"\|_\infty$ when $\lambda = 1$ (left) and $\lambda = 10$ (right) for both approaches (Proposition 5.1 and 5.2).

5.3 Upper-Bound for the cost function (Proof of i) of Theorem 2.3)

Remind that we assume that there are two local minima for U denoted by $x_1^\star$ and $x_2^\star$ with $U(x_1^\star) < U(x_2^\star)$ and a local maximum denoted by $x^\star$. We set again $z_1^\star = (x_1^\star, 0)$, $z_2^\star = (x_2^\star, 0)$ and $z^\star = (x^\star, 0)$. In this particular setting, we want to obtain an upper-bound for $I(z_2^\star, z_1^\star)$ and then for W. This is the purpose of the next proposition.

Proposition 5.3 *Assume that U is a one-dimensional double well potential defined as above such that $U(x_1^\star) < U(x_2^\star)$. Then, for all $\lambda > 0$,*

$$W(z_1^\star) = I(z_2^\star, z_1^\star) \leq 2(U(x^\star) - U(x_2^\star)).$$

where $z_1^\star = (x_1^\star, 0)$, $z_2^\star = (x_2^\star, 0)$ and $x^\star$ denotes the unique local maximum of U.

Essentially, the previous proposition is a consequence of the following Lemma 5.1 and Lemma 5.2 combined with the fact that $I(z_2^\star, z_1^\star) \leq I(z_2^\star, z^\star) + I(z^\star, z_1^\star)$. Let us stress that the proofs of Lemma 5.1 and Lemma 5.2 rely both on Lemma 5.3.

Lemma 5.1 *Under the assumptions of Proposition 5.3, we have*

$$I(z^\star, z_i^\star) = 0 \quad \text{for } i = 1, 2.$$

<u>*Proof :*</u> We show the result for $i = 1$. The last part of the proof of this Lemma relies on Lemma 5.3. Indeed, first, we prove that for every initial point $z_\varepsilon = (x_\varepsilon, y_\varepsilon)$ with $U(x_\varepsilon) + |y_\varepsilon|^2/2 < U(x^\star)$ and $x_\varepsilon \in [x_1^\star, x^\star]$, $I(z_\varepsilon, z_1^\star) = 0$. Second, applying Lemma 5.3 we will have $I(z^\star, z_\varepsilon) \leq \varepsilon$, for all $\varepsilon > 0$ and taking ε goes to zero the result follows.

In order to prove the first part, we adopt a similar strategy as the proof used in Section 3.4. Indeed, assume that $(\mathbf{x}(t), \mathbf{y}(t))$ is solution of

$$\begin{cases} \dot{\mathbf{x}}(t) = -\mathbf{y}(t) \\ \dot{\mathbf{y}}(t) = \lambda(U'(\mathbf{x}(t)) - \mathbf{y}(t)) \end{cases} \tag{5.11}$$

starting from z_ε. Considering the function F defined by $F(t) = \mathcal{E}(\mathbf{x}(t), \mathbf{y}(t)) = U(\mathbf{x}(t)) + |\mathbf{y}(t)|^2/(2\lambda)$, one can check that $F'(t) = -\mathbf{y}(t)^2$. In particular, F is a positive non-increasing function thus convergent. Then, $(\mathbf{x}(t), \mathbf{y}(t))_{t\geq 0}$ is bounded and the fact that $\mathcal{E}(z_\varepsilon) \leq U(x^\star)$ shows that z_ε belongs to a compact set, as well as all the trajectories initialized in $\mathcal{E}^{-1}(]-\infty; U(x^\star)])$. Since U" is continuous, we then deduce that U' is Lipschitz continuous on the set where $(\mathbf{x}, \mathbf{y})$ is living. It implies classically that the family of shifted trajectories $(\mathbf{z}(t + .))_{t\geq 0}$ is relatively compact for the topology of uniform convergence on compact sets. Now, let $\mathbf{z}^\infty$ denote the limit of a convergent subsequence $(\mathbf{z}(t_n + .))_{n\geq 0}$, $\mathbf{z}^\infty$ is a solution of (5.11) and since F is continuous and converges as $t \to +\infty$ to some limit l, we have necessarily $\mathcal{E}(\mathbf{x}^\infty(s), \mathbf{y}^\infty(s)) = l$ for all $s \geq 0$. Thus we get

$$\frac{d}{dt}\mathcal{E}(\mathbf{x}^\infty(t), \mathbf{y}^\infty(t)) = 0,$$

for all $t \geq 0$. Using that $F'(t) = -\mathbf{y}(t)^2$ for a solution of (5.11), we then obtain that $\mathbf{y}^\infty(t) = 0$ for every $t \geq 0$.

Thus, $\mathbf{x}^\infty$ is constant and $\mathbf{z}^\infty$ is a *stationary* solution of the ordinary differential equation (5.11). We can deduce that every accumulation point of $(\mathbf{x}(t), \mathbf{y}(t))$ belongs to $\{(x, y) \in \mathbb{R}, b(x, y) = 0\}$. Under the assumption $U(x_\varepsilon) + |y_\varepsilon|^2/2 < U(x^\star)$, and since F is non increasing, the only possible accumulation point is $z_1^\star$. Then, $(\mathbf{x}(t), \mathbf{y}(t)) \xrightarrow{t\to+\infty} z_1^\star$ and $I(z_\varepsilon, z_1^\star) = 0$. As a consequence, we have

$$I(z^\star, z_1^\star) \leq I(z^\star, z_\varepsilon).$$

We shall now prove the second part. As announced at the beginning of the proof, applying Lemma 5.3, we get that for every $\varepsilon > 0$, one may find $z_\varepsilon = (x_\varepsilon, y_\varepsilon)$ such that we both have $U(x_\varepsilon) + |y_\varepsilon|^2/2 < U(x^\star)$, $x_\varepsilon \in [x_1, x^\star]$ and $I(z^\star, z_\varepsilon) \leq \varepsilon$. Taking finally $\varepsilon \to 0$, the result holds □

Lemma 5.2 *Assume the assumptions of Proposition 5.3. Then,*

$$I(z_2^\star, z^\star) \leq 2(U(x^\star) - U(x_2^\star)).$$

Proof : The idea of the proof is to use the "reverse" differential flow (see (5.12) below). Since $z_2^\star$ is an equilibrium point of $\dot{\mathbf{z}} = b(\mathbf{z})$, it follows from Lemma 5.3 that for all $\varepsilon > 0$, there exists $\tilde{z}_\varepsilon$ such that $\tilde{z}_\varepsilon = z_2^\star + \delta_\varepsilon(z^\star - z_2^\star) = (x_2^\star + \delta_\varepsilon(x^\star - x_2^\star), 0)$ with $\delta_\varepsilon \in (0,1)$, $U(\tilde{x}_\varepsilon) > U(x_2^\star)$ and $I(z_2^\star, \tilde{z}_\varepsilon) \leq \varepsilon$. This implies that it is now enough to prove that

$$\forall\, \varepsilon > 0, \quad I(\tilde{z}_\varepsilon, z^\star) \leq 2(U(x^\star) - U(\tilde{x}_\varepsilon)).$$

Let us consider $\mathbf{z}_\varepsilon = (\mathbf{x}_\varepsilon, \mathbf{y}_\varepsilon)$ defined as the solution of

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{y}(t) = -\mathbf{y}(t) + 2\mathbf{y}(t) \\ \dot{\mathbf{y}}(t) = \lambda(U'(\mathbf{x}(t)) - \mathbf{y}(t)) \end{cases} \tag{5.12}$$

starting from $\tilde{z}_\varepsilon$. Note that setting $\phi(t) = 2\int_0^t \mathbf{y}(s)ds$, $\mathbf{z}_\varepsilon = \mathbf{z}_\varphi$. Let us study its asymptotic behavior. To this end, we introduce now the function $\tilde{F}$ defined by $\tilde{F}(t) = \frac{\mathbf{y}_\varepsilon(t)^2}{2\lambda} - U(\mathbf{x}_\varepsilon(t))$ ($\tilde{F}$ is the equivalent of function F defined above for the uncontrolled trajectory). We observe that $\tilde{F}'(t) = -\mathbf{y}_\varepsilon(t)^2$ and thus that $\tilde{F}$ is non-increasing.

We first show that the solution $(\mathbf{x}_\varepsilon(t), \mathbf{y}_\varepsilon(t))$ to (5.12) starting from $\tilde{z}_\varepsilon$ satisfies necessarily $x_1^\star < \mathbf{x}_\varepsilon(t) < x_2^\star$ for all $t \geq 0$. Actually, on the one hand, let $\tau_1 := \inf\{t \geq 0, \mathbf{x}_\varepsilon(t) = x_1^\star\}$. Then, if $\tau_1 < +\infty$, remark that $\tilde{y}_\varepsilon = 0$ and one should have

$$\frac{\mathbf{y}_\varepsilon(\tau_1)^2}{2\lambda} - U(x_1^\star) = -U(\tilde{x}_\varepsilon) - \int_0^{\tau_1} \mathbf{y}_\varepsilon(s)^2 ds \tag{5.13}$$

and it would follow that $U(\tilde{x}_\varepsilon) - U(x_1^\star) < 0$, which is impossible under our assumptions. Thus, $\tau_1 = +\infty$. On the other hand, the fact that $U(\tilde{x}_\varepsilon) > U(x_2^\star)$ implies in a same way that $\tau_2 := \inf\{t \geq 0, \mathbf{x}_\varepsilon(t) = x_2^\star\}$ satisfies $\tau_2 = +\infty$. Thus, we obtain that $(\mathbf{x}_\varepsilon(t))_{t\geq 0}$ belongs to the interval $(x_1^\star, x_2^\star)$.

This point combined with the decreasing property of $\tilde{F}$ implies that

$$\sup_{t\geq 0} |\mathbf{y}_\varepsilon(t)|^2 \leq 2\lambda \left(\tilde{F}(0) + \sup_{x\in[x_1,x_2]} U(x) \right) < +\infty.$$

As a consequence, $(\mathbf{x}_\varepsilon(t), \mathbf{y}_\varepsilon(t))_{t\geq 0}$ is bounded and a similar argument as in the proof of Lemma 5.1 yields that the limit $(\mathbf{x}_\varepsilon^\infty, \mathbf{y}_\varepsilon^\infty)$ of any convergent subsequence $(\mathbf{x}_\varepsilon(t_n + .), \mathbf{y}_\varepsilon(t_n + .))_{n\geq 1}$ lies in the stationary solutions of (5.12). Thus, we deduce that $\mathbf{x}_\varepsilon^\infty \in \{x_1^\star, x^\star, x_2^\star\}$.

Now, Equation (5.13) and the fact that $U(\tilde{x}_\varepsilon) > U(x_1^\star)$ and $U(\tilde{x}_\varepsilon) > U(x_2^\star)$ imply that $\mathbf{x}_\varepsilon^\infty$ can not be $x_1^\star$ or $x_2^\star$. Thus,

$$(\mathbf{x}_\varepsilon(t), \mathbf{y}_\varepsilon(t)) \xrightarrow{t\to+\infty} z^\star.$$

Finally, since $\mathbf{z}_\varepsilon = \mathbf{z}_\varphi$ with $\dot{\varphi} = 2\mathbf{y}_\varepsilon$, we deduce that

$$I(\tilde{z}_\varepsilon, z^\star) \leq \frac{1}{2}\int_0^{+\infty} |\dot{\varphi}(s)|^2 ds = 2\int_0^{+\infty} |\mathbf{y}_\varepsilon(s)|^2 ds = 2(U(x^\star) - U(\tilde{x}_\varepsilon)).$$

The announced result follows. $\square$

Lemma 5.3 *Let x_0 denote an equilibrium point for U such that, in the neighbourhood of x_0, U is strictly convex (resp. strictly concave) if x_0 is a local maximum (resp. a local minimum). Let $v \in \mathbb{R}^2$ with $|v| = 1$ and set $z_0 = (x_0, 0)$. Then, for all positive ε and ρ, there exists $\tilde{\rho} > 0$ and $\tau \geq 0$ such that $z_\varepsilon := z_0 + \tilde{\rho}v$ satisfies $I_\tau(z_0, z_\varepsilon) \leq \varepsilon$.*

Note that this point is a direct consequence of Lemma 4.1 if $U''(x_0) \neq 0$. However, we choose to give below an alternative proof under a less constraining assumption, based on the turning dynamics of the system.

Proof : Set $z_0 = (x_0, 0)$. Since $b(x_0, 0) = 0$, we first remark that for every $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$, such that the solution $(\mathbf{z}_1(t))_{t\geq 0}$ of the controlled trajectory (with control $\varphi = 1$)

$$\begin{cases} \dot{\mathbf{x}}(t) = -\mathbf{y}(t) + 1 \\ \dot{\mathbf{y}}(t) = \lambda(U'(\mathbf{x}(t)) - \mathbf{y}(t)) \end{cases}$$

satisfies $\mathbf{x}(\delta_\varepsilon) \neq x_0$, $|\mathbf{z}_1(\delta_\varepsilon) - z_0| \leq \varepsilon$ and $I(z_0, \mathbf{z}_1(\delta_\varepsilon)) \leq \varepsilon$. Let us now consider separately the cases where x_0 is either a local minimum or a local maximum. In the first case, we consider the solution $(\mathbf{z}_2(t))_{t\geq\delta_\varepsilon}$ to (5.11) satisfying $\mathbf{z}_2(\delta_\varepsilon) = \mathbf{z}_1(\delta_\varepsilon)$. We already know from the proof of Lemma 5.1 that $(\mathbf{z}_2(t))_{t\geq\delta_\varepsilon}$ converges to z_0 for ε small enough (this part of the proof of Lemma 5.1 did not rely on the result of Lemma 5.3). Using that U is strictly convex in the neighbourhood of x_0, if we let δ_ε small enough, we also deduce from Theorem 6.1 of Cabot, Engler, and Gadat (2009a) that the number of sign changes (oscillations of the dynamical system) of $(\dot{\mathbf{x}}(t))$ and thus of $(\mathbf{y}(t))$ is infinite. These two points imply that $(\mathbf{z}_2(t))$ converges to z_0 by turning around z_0 and for every $v \in \mathbb{R}^2$ with $|v| = 1$, for every $\rho > 0$, there exist $\tau \geq \delta_\varepsilon$ and $\tilde{\rho} \in (0, \rho)$ such that $\mathbf{z}_2(\tau) = x_0 + \tilde{\rho}v := z_\varepsilon$. The result follows in this case using that $I_\tau(z_0, \mathbf{z}_2(\tau)) \leq I_{\delta_\varepsilon}(z_0, \mathbf{z}_1(\delta_\varepsilon)) \leq \varepsilon$.

Consider now the case where x_0 is a local maximum, the proof is similar but $(\mathbf{z}_2(t))_{t\geq\delta_\varepsilon}$ is replaced by the solution of (5.12) starting from $\mathbf{z}_1(\delta_\varepsilon)$ and U by $-U$. Following the proof of Lemma 5.2, we obtain in this case that,

$$\begin{aligned} I(z_0, \mathbf{z}_2(\tau)) &\leq I_{\delta_\varepsilon}(z_0, \mathbf{z}_1(\delta_\varepsilon)) + 2\int_{\delta_\varepsilon}^{\tau} |\mathbf{y}_2(s)|^2 ds \\ &\leq \varepsilon + 2\int_{\delta_\varepsilon}^{+\infty} |\mathbf{y}(s)|^2 ds \leq \varepsilon + 2\left(U(x_0) - U(\mathbf{x}_1(\delta_\varepsilon)) - \frac{|\mathbf{y}_1(\delta_\varepsilon)|^2}{2}\right) \end{aligned}$$

and the result follows taking δ_ε small enough. □

Appendix A: Proof of Proposition 3.2

Let $\varepsilon > 0$ and h be a bounded continuous function. Since ν_ε is an invariant distribution, we have for all $t > 0$,

$$\int h^{\frac{1}{\varepsilon^2}} d\nu_\varepsilon = \int h_{\varepsilon,t} d\nu_\varepsilon \quad \text{where} \quad h_{\varepsilon,t}(z) = \mathbb{E}[h^{\frac{1}{\varepsilon^2}}(Z_t^{\varepsilon,z})].$$

Since h is bounded continuous, it follows from Assumption (ii) and Lemma 3.1.12 of (Puhalskii, 2001) that for all $z \in \mathbb{R}^{2d}$, for all $(z_\varepsilon)_{\varepsilon>0}$ such that $z_\varepsilon \to z$,

$$\lim_{\varepsilon\to 0} (h_{\varepsilon,t})^{\varepsilon^2}(z_\varepsilon) = \sup_{v\in\mathbb{R}^{2d}} h(v)\exp(-I_t(z, v)). \tag{A.1}$$

Now, since $(\nu_\varepsilon)_{\varepsilon>0}$ is exponentially tight, $(\nu_\varepsilon)_{\varepsilon>0}$ admits some (LD)-convergent subsequence. Let $(\nu_{\varepsilon_n})_{n\geq 1}$ denote such a subsequence. Then, $(\nu_{\varepsilon_n})_{n\geq 1}$ satisfies a large deviation principle with speed ε^{-2} and rate function denoted by W. Then, by Lemma 3.1.12 of (Puhalskii, 2001), we have

$$\left(\int h^{\frac{1}{\varepsilon_n^2}} d\nu_{\varepsilon_n}\right)^{\varepsilon_n^2} \xrightarrow{n\to+\infty} \sup_{z\in\mathbb{R}^{2d}} h(z)\exp(-W(z))$$

and by (A.1) and Lemma 3.1.13 of (Puhalskii, 2001), we obtain that

$$\left(\int h_{\varepsilon_n,t} d\nu_{\varepsilon_n}\right)^{\varepsilon_n^2} \xrightarrow{n\to+\infty} \sup_{z\in\mathbb{R}^{2d}} \left(\sup_{v\in\mathbb{R}^{2d}} h(v)\exp(-I_t(z, v))\exp(-W(z))\right).$$

It follows that for all bounded continuous function h,

$$\sup_{z\in\mathbb{R}^{2d}} h(z)\exp(-W(z)) = \sup_{z\in\mathbb{R}^{2d}} h(z)\left(\sup_{v\in\mathbb{R}^{2d}} \exp(-I_t(v,z))\exp(-W(v))\right).$$

By Theorem 1.7.27 of (Puhalskii, 2001), the above equality holds in fact for all bounded measurable function h. Applying this equality with $h=\mathbf{1}_{\{z_0\}}$

$$\exp(-W(z_0)) = \sup_{v\in\mathbb{R}^{2d}} \exp(-I_t(v,z_0))\exp(-W(v)),$$

and the result follows. $\square$

Appendix B: Proof of Lemma 3.2

The explicit computation of $\mathcal{A}^\varepsilon V^p$ gives for all (x,y),

$$\begin{aligned}\mathcal{A}^\varepsilon V^p(x,y) &= pV^{p-1}(x,y)\Big(-m\langle x,\nabla U(x)\rangle-(1-m)|y|^2\Big)\\ &\quad+\frac{\varepsilon^2}{2}\mathrm{Tr}\Big[p(p-1)V^{p-2}\nabla_x V\otimes\nabla_x V+pV^{p-1}D_x^2V\Big],\end{aligned} \tag{B.1}$$

where for $u,v\in\mathbb{R}^d$, $u\otimes v$ is the $d\times d$ matrix defined by $(u\otimes v)_{i,j}=u_iv_j$.

Then, let us prove (3.9) under Assumption $(\mathbf{H_{Q+}})$. Since $m\in(0,1)$, we have

$$-m\langle x,\nabla U(x)\rangle-|y|^2(1-m)\leq m\beta-m\alpha U(x)-(1-m)|y|^2\leq\beta_1-\alpha_1V(x,y),$$

for some constants $\beta_1\in\mathbb{R}$ and $\alpha_1>0$. Moreover, since $D_x^2V(x,y)=D^2U(x)+mI_d$, $\rho\in(0,1)$ and $\lim_{|(x,y)|\to+\infty}V(x,y)=+\infty$ (see (3.16)), we have

$$\frac{Tr\Big[p(p-1)V^{p-2}\nabla_xV\otimes\nabla_xV+pV^{p-1}D_x^2V\Big]}{V^p(x,y)}\longrightarrow 0\quad\text{as } |(x,y)|\longrightarrow+\infty.$$

It follows that there exists $\beta_2>0$ such that for all $\varepsilon\in(0,1]$,

$$\frac{\varepsilon^2}{2}Tr\Big[p(p-1)V^{p-2}\nabla_xV\otimes\nabla_xV+pV^{p-1}D_x^2V\Big]\leq\beta_2+\frac{p\alpha_1}{2}V^p(x,y).$$

Therefore, we get for all $\varepsilon\in(0,1]$

$$\mathcal{A}V^p(x,y)\leq p\beta_1V^{p-1}(x,y)+\beta_2-\frac{p\alpha_1}{2}V^p(x,y).$$

Using again that $\lim_{|(x,y)|\to+\infty}V(x,y)=+\infty$, we deduce that $p\beta_1V^{p-1}\leq\beta_3+\frac{p\alpha_1}{4}V^p$ and we deduce there exist some positive $\tilde\beta$ and $\tilde\alpha$ such that

$$\forall\varepsilon\in(0,1],\ \forall(x,y)\in\mathbb{R}^{2d},\quad\mathcal{A}V^p(x,y)\leq\tilde\beta-\tilde\alpha V^p(x,y).$$

Let us now consider (3.9) under assumption $(\mathbf{H_{Q-}})$. Here, we fix $p\in(a-1,a)$. Since $m\in(0,1)$, we have that

$$-m\langle x,\nabla U(x)\rangle-|y|^2(1-m)\leq m\beta-m\alpha(|x|^2)^a-(1-m)|y|^2\leq\beta_1-\alpha_1V^a(x,y)\quad(\beta'\in\mathbb{R},\alpha'>0)$$

where in the second inequality, we used the elementary inequalities $u^{2a}\leq 1+u^2$ for $u\geq 0$ and $(u+v)^a\leq u^a+v^a$ for $u,v\geq 0$, and the fact that $V(x,y)\leq C(1+|x|^2)+|y|^2)$ $(|\nabla U|^2\leq C(1+U)$

implies that $\sqrt{U}$ is sublinear).
Under $(\mathbf{H_{Q-}})$ we also have

$$\sup_{x\in\mathbb{R}^d} \|D_x^2 V(x,y)\| < +\infty,$$

and since $p \in 0, 1$ we have

$$\mathrm{Tr}\Big[p(p-1)V^{p-2}\nabla_x V \otimes \nabla_x V + pV^{p-1}D_x^2 V\Big] \quad \leq \quad CV^{p-1}(x,y).$$

This way, there exist $\tilde{\alpha} > 0, \tilde{\beta}$ such that for all $\varepsilon \in [0,1]$ and all (x,y), we have

$$\mathcal{A}V^p(x,y) \leq \beta - \alpha V^{p+a-1}(x,y).$$

Now we prove inequality (3.10). One can check that

$$\mathcal{A}^\varepsilon \psi_\varepsilon = \frac{\delta}{\varepsilon^2}\psi_\varepsilon\Bigg(- pV^{p-1}\Big(m\langle x, \nabla U(x)\rangle + (1-m)|y|^2\Big) \tag{B.2}$$

$$+ \frac{1}{2}\mathrm{Tr}\left[\varepsilon^2\left(p(p-1)V^{p-2} + \delta p^2 V^{2p-2}\right)\nabla_x V \otimes \nabla_x V + \varepsilon^2 pV^{p-1}D_x^2 V\right]\Bigg). \tag{B.3}$$

We recall that $\nabla_x V = \nabla U + m(x-y)$ and that $D_x^2 V = D^2 U + mI_d$. Thus, using $(\mathbf{H_{Q+}})(ii)$ and $(\mathbf{H_{Q-}})(ii)$, we obtain that when $|(x,y)| \to +\infty$,

$$(\text{B.3}) = \begin{cases} F_1 + F_2 & \text{under } (\mathbf{H_{Q+}}) \\ F_1 & \text{under } (\mathbf{H_{Q-}}), \end{cases}$$

where $F_1 \leq C(1 + V^{2p-1})$ for a suitable constant C and $F_2/V^p \to 0$ as $|(x,y)| \to +\infty$. Then, since $2p-1 < p$ if $p \in (0,1)$ and that $2p-1 < p+a-1$ if $p < a$, we obtain easily (3.10) by following the lines of the part of the proof. $\square$

References

Alvarez, F. (2000). On the minimizing property of a second order dissipative system in hilbert spaces. *SIAM J. Control Optim.*, *38*, 110–1119.

Azencott, R. (1980). *Large deviations theory and applications* (Vol. 774). Springer-Verlag.

Barles, G. (1994). *Solutions de viscosité des équations de hamilton-jacobi* (Vol. 17). Springer-Verlag.

Benaim, M. (1996). A dynamical system approach to stochastic approximations. *SIAM J. Control Optim.*, *34*(2), 437–472.

Biswas, A., & Borkar, V. S. (2009). Small noise asymptotics for invariant densities for a class of diffusions: a control theoretic view. *J. Math. Anal. Appl.*, *360*(2), 476–484.

Biswas, A., & Budhiraja, A. (2011). Exit time and invariant measure asymptotics for small noise constrained diffusions. *Stochastic Process. Appl.*, *121*(5), 899–924.

Cabot, A. (2009). Asymptotics for a gradient system with memory term. *Proc. of the American Mathematical Society*, *9*, 3013–3024.

Cabot, A., Engler, H., & Gadat, S. (2009a). On the long time behavior of second order differential equations with asymptotically small dissipation. *Trans. of the American Mathematical Society*, *361*, 5983–6017.

Cabot, A., Engler, H., & Gadat, S. (2009b). Second-order differential equations with asymptotically small dissipation and piecewise flat potentials. *Electronic Journal of Differential Equation*, *17*, 33–38.

Cerrai, S., & Röckner, M. (2005). Large deviations for invariant measures of stochastic reaction-diffusion systems with multiplicative noise and non-Lipschitz reaction term. *Ann. Inst. H. Poincaré Probab. Statist.*, *41*(1), 69–105.

Chiang, T.-S., Hwang, C.-R., & Sheu, S. J. (1987). Diffusion for global optimization in R^n. *SIAM J. Control Optim.*, *25*(3), 737–753.

Coron, J.-M. (2007). *Control and nonlinearity* (Vol. 136). Providence, RI: American Mathematical Society.

Dembo, A., & Zeitouni, O. (2010). *Large deviations techniques and applications* (Vol. 38). Berlin: Springer-Verlag. (Corrected reprint of the second (1998) edition)

Feng, J., & Kurtz, T. G. (2006). *Large deviations for stochastic processes* (Vol. 131). Providence, RI: American Mathematical Society.

Freidlin, M. I., & Wentzell, A. D. (1979). *Random perturbations of dynamical systems* (Second ed., Vol. 260). New York: Springer-Verlag. (Translated from the 1979 Russian original by Joseph Szücs)

Gadat, S., & Panloup, F. (2012). Long time behaviour and stationary regime of memory gradient diffusions. *Annales de l'Institut Henri Poincaré, to appear*, 1–36.

Has'minskii, R. (1960). Ergodic properties of recurrent diffusion processes and stabilization of the solution to the cauchy problem for parabolic equations. *Theory of probability and its applications*, *5*, 179 –196.

Herrmann, S., & Tugaut, J. (2010). Stationary measures for self-stabilizing processes: asymptotic analysis in the small noise limit. *Electron. J. Probab.*, *15*, no. 69, 2087–2116.

Holley, R., & Stroock, D. (1988). Simulated annealing via Sobolev inequalities. *Comm. Math. Phys.*, *115*(4), 553–569.

Khas'minskiĭ, R. (1980). *Stochastic stability of differential equations.* Sijthoff & Noordhoff, Alphen aan den Rijn (The Nederlands).

Miclo, L. (1992). Recuit simulé sur $\mathbf{R}^n$. Étude de l'évolution de l'énergie libre. *Annales de l'Institut Henri Poincaré. Probabilités et Statistiques*, *28*(2), 235–266.

Puhalskii, A. (2001). *Large deviations and idempotent probability* (Vol. 119). Chapman & Hall/CRC, Boca Raton, FL.

Puhalskii, A. (2003). On large deviation convergence of invariant measures. *J. Theoret. Probab.*, *16*(3), 689–724.

Puhalskii, A. (2004). On some degenerate large deviation problems. *Electron. J. Probab.*, *9*, no. 28, 862–886. Available from `http://dx.doi.org/10.1214/EJP.v9-232`

Raimond, O. (2009). Self-interacting diffusions: a simulated annealing version. *Probab. Theory Related Fields*, *144*(1-2), 247–279.

Stroock, D., & Varadhan, S. R. S. (1970). On the support of diffusion processes with applications to the strong maximum principle. *Symposium of Math. Statist. Prob. III*, 333–359.

Stroock, D., & Varadhan, S. R. S. (1972). On degenerate elliptic-parabolic operators of second order and their associated diffusions. *Comm. Pure Appl. Math.*, *25*, 651–713.